Math 412/512 Assignment 5

Due Thursday, November 17

1) Show that the following group-subgroup inclusions are normal.

a) $\mathbb{Q} \leq \mathbb{R}$.

b) $H = \langle R_{180} \rangle \leq D_4$

c) $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$.

2) Show that $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{Q}^\times \right\} \leq GL_n(\mathbb{Q})$ is not normal.

3) Let $H \leq G$.

a) Suppose that $|G| < \infty$. Prove that the number of left cosets of H in G is equal to the number of right cosets of H in G.

b) Suppose $|G| = \infty$. Let $\phi : G \to G$, $\phi(g) = g^{-1}$. Show that $\phi(gH) = Hg^{-1}$ and conclude that ϕ is a bijection from the set of left cosets of G to the set of right cosets of G.

c) Determine that $[G : H]$ is unchanged whether one uses left or right cosets in the definition.

d) (# 7, Chapter 9) If $[G : H] = 2$, prove that $H \triangleleft G$. Which group must G/H then be isomorphic to?

4)(# 28, Chapter 9) Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, $H = \{(0,0), (2,0), (0,2), (2,2)\}$ and $K = \langle (1,2) \rangle$.

a) Is G/H isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?

b) Is G/K isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?

5) Let $H \triangleleft G$. Prove that G/H is cyclic if and only if there is an element $g \in G$ with the following property: for every $k \in G$, there is some integer n such that $kg^n \in H$.

Extra Credit Problems

Directions: I will accept no written solutions for even a part of the following problems. They must be proved on a blackboard with me listening to the proof.

1) $H \leq G$ is called *characteristic* in G if for all $\phi \in Aut(G)$, $\phi(H) = H$. Note that, by a problem from class, $\phi_g(H) = H$ for all inner automorphisms ϕ_g which implies that a characteristic subgroup is automatically normal.

a) Show that if $K \leq H \leq G$, $H \triangleleft G$, and K is characteristic in H, then $K \triangleleft G$.

b) Show that every subgroup of a cyclic group is characteristic.

2) Let $G = GL_2(\mathbb{R})$ and $H \leq G$ consist of all $A \in G$ with $det(A) > 0$. Find a familiar group isomorphic to G/H.