Math 412/512 Assignment 6

Due Tuesday, November 29

1) Complete the proof, begun in class, that $C(\mathbb{R})$ is a ring by checking that the distributive property holds.

2) Let G be an abelian group, $|G| > 1$. The *trivial ring structure* on G is given by endowing G with the multiplication

$$g \cdot_T h = e_G$$

for all $g, h \in G$. By definition, " $\cdot_T$ " is a binary operation.

a) Check that "$\cdot_T$" is associative and distributes over the group operation of G, which you may write as " $+$ ".

Thus, equipping G with its group operation " $+$ " and the multiplication " $\cdot_T$ " makes G into a ring.

b) Is G a commutative ring?

c) Does G have a unity?

3) Let $S = \{f : \mathbb{Z} \to \mathbb{Z} \mid f \text{ is bijective}\}$. Define two binary operations, "$+_S$" and " $\cdot_S$ " by

1. $(f +_S g)(n) = f(n) + g(n)$
2. $(f \cdot_S g)(n) = (f \circ g)(n)$

for all $f, g \in S$ and all $n \in \mathbb{Z}$. Check that, with these two operations, S is NOT a ring.

4) In the following problems, show that the subset is a subring. You may assume, in each example, that the larger set is a ring.

a) $\mathbb{Z}[\sqrt{3}] := \{x \in \mathbb{R} |\ x = a + b\sqrt{3},\ a, b \in \mathbb{Z}\} \subset \mathbb{R}$

b) $T_3 := \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{pmatrix} |a_{i,j} \in \mathbb{R}\ \forall\ 1 \le i \le j \le 3 \right\} \subset M_3(\mathbb{R})$.

c) $I_0 := \{f \in C(\mathbb{R}) \mid f(0) = 0\} \subset C(\mathbb{R})$.

5) (# 36, Chapter 12) Let m and n be positive integers and let k be the least common multiple of m and n. Show that $m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$.

Extra Credit Problem

Directions: I will accept no written solutions for even a part of the following problem. They must be proved on a blackboard with me listening to the proof.

1) (group rings) Let G be a finite group and let $\mathbb{C}[G]$ be the space of all functions from G to $\mathbb{C}$. $\mathbb{C}[G]$ is a finite-dimensional vector space over $\mathbb{C}$ with basis consisting of the functions $\delta_g : G \to \mathbb{C}$,

$$\delta_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g \end{cases}.$$

As such, if $f_1, f_2 \in \mathbb{C}[G]$, there are complex number $(\alpha_k)_{k \in G}$ and $(\beta_h)_{h \in G}$ with

$$f_1 = \sum_{k \in G} \alpha_k \delta_k, \ f_2 = \sum_{h \in G} \beta_h \delta_h.$$

Note that these are finite sums since G is finite.

Define " $+_G$ " and " $\cdot_G$ " by

1. $(f_1 +_G f_2)(g) = f_1(g) + f_2(g)$
2. $(f_1 \cdot_G f_2)(g) = \displaystyle\sum_{k,h \in G} \alpha_k \beta_h \delta_{kh}(g)$.

a) Show that $\mathbb{C}[G]$, with these operations, is a ring.

b) Prove that if $\sigma = \displaystyle\sum_{g \in G} \delta_g$, then for all $f \in \mathbb{C}[G]$, $f \cdot_G \sigma = \sigma \cdot_G f$.