Math 412/512 Assignment 1

1) Complete graphs are a fundamental object in graphs theory. I recommend sharpening your induction skills on it, but if you can do it another way, I'd like to see that!

(#10, Chapter 1) A **complete graph** is a collection of n points, each of which is connected to each other point. The complete graphs for on 3,4, and 5 points are given at the bottom of page 9 in the text. Prove that the complete graph on n points has exactly $n(n-1)/2$ lines.

2) This question collects some of the most useful facts concerning functions and tests your familiarity with the terms "injective," "bijective," and "surjective".

Let S, T, and U be sets and let $f : T \to U$, $g : S \to T$.

a) Prove that if $f \circ g$ is bijective, then f is surjective and g is injective.

b) Verify that if S is finite, then any injection from S to itself is also surjective. This result sometimes goes by the fancy name "pigeonhole principle".

c) If f is surjective and g is injective, does it then follow that $f \circ g$ must be bijective? Prove or give a counterexample.

d) Give an example of an injection $f : \mathbb{N} \to \mathbb{N}$ that is not surjective. You just need an example here, not a proof.

3) This kind of divisibility statement is the essential part of the proof that there are infinitely many prime numbers. Whether there are infinitely many twin prime numbers is an open question! Prove that if $m, n \in \mathbb{Z}$ and $n > 1$, then if n divides m, n does not divide $m^2 + 1$.

4) I can't even remember what this is supposed to show, but my guess is that the square root of two is irrational

(#15, Chapter 2) See the picture and explanation on page 22 of the text.

a) Find the point E by intersecting the side CD with the perpendicular to the diagonal AC at P. First show that the length of segment EC is $\sqrt{2}r$.

b) Prove that the length of segment DE is r, by showing that the triangle DEP is an isosceles triangle. Why does this mean that the next step in Euclid's algorithm yields $s = 2r + (\sqrt{2} - 1)r$?

c) Argue that the next step of the algorithm yields

$$r = 2(\sqrt{2} - 1)r + (\sqrt{2} - 1)^2 r.$$

Conclude that this algorithm never halts, and so there is no common measure for the diagonal and side of the square.