Math 412/512 Assignment 3

Due Thursday, October 15

1) (#15, Chapter 5) a) Apply Eisenstein's Criterion to check that the following polynomials are irreducible:

$$5x^3 - 6x^2 + 2x - 14 \text{ and } 4x^5 + 5x^3 - 15x + 20.$$

b) Prove that if $f(x) \in \mathbb{Z}[x]$, then $f(x)$ is irreducible if and only if $f(y+m)$ is.

2) (#16, Chapter 5) Let p be a positive prime and let

$$\Phi_p[x] = \frac{x^p - 1}{x - 1}.$$

Prove that $\Phi_p[x]$ is irreducible over $\mathbb{Z}[x]$, using Eisenstein's Criterion and part b) of **1)** with $m = 1$.

3) In $M_2(\mathbb{C})$, let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Verify that $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$, and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -I$.

4) (#18, Chapter 6) With the notation $a^2 = a \cdot a$, suppose that $a^2 = a$ for every element a in a ring R. (Elements a in a ring R where $a^2 = a$ are called **idempotent**.)

a) Show that $a = -a$.

b) Now show that R is commutative.

5) In the following problems, determine whether the indicated subset is a subring. You may assume, in each example, that the larger set is a ring.

a) $\mathbb{Z}[\sqrt{5}] := \{x \in \mathbb{R} | \ x = a + b\sqrt{5}, \ a, b \in \mathbb{Z}\} \subset \mathbb{R}$

b) $N_3 := \{T \in M_3(\mathbb{R}) \mid \exists\, 0 \neq S \in M_3(\mathbb{R}), \ ST = 0\} \subset M_3(\mathbb{R})$.

c) Referring to Problem **3)** above,

$$\mathbb{H} = \{aI + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, \text{ and } d \in \mathbb{R}\} \subset \mathrm{M}_2(\mathbb{C})$$

6) (#20, Chapter 7) Suppose that R is a set with two operations $+$ and $\circ$, which satisfy the rules defining a ring, except we do not assume that addition is commutative. Suppose that R also has a multiplicative identity 1. Then prove that addition in R must in fact be commutative, and so R is a ring under these operations.

7) (#19, Chapter 8) Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$$

is a nonzero element which is not a unit. Show that A is actually a zero divisor. (You might want to look at Exercise #2 in Chapter 8). This is in stark contrast to $\mathbb{Z}$, where the only units are ± 1 and there are NO zero divisors!

Extra Credit: You may have seen in advanced calculus that $\mathbb{R}$ and $\mathbb{C}$ have the same cardinality. Prove that $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic as rings. I will accept no written solution, you must present your solution to me on the board in my office.