Math 412/512 Assignment 4

Due Thursday, October 22

1) Recall that $C_0(\mathbb{R})$ is the ring of all continuous function $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{|x|\to\infty} f(x) = 0$. Fix a real number x and define

$$I_x = \{f \in C_0(\mathbb{R}) \mid f(x) = 0\}.$$

a) Show that, for any choice of x, I_x is an ideal in $C_0(\mathbb{R})$.

b) Is I_x a principal ideal? Prove or disprove.

2) (# 25, Chapter 9) Suppose that R is a commutative ring with unity, and I is an ideal. Let

$$A(I) = \{s \in R \ : \ rs = 0 \ \forall \ r \in I\};$$

we call $A(I)$ the **annihilator** of I. Prove that $A(I)$ is an ideal.

3) (#'s 12 and 14, Chapter 10) In this exercise, we describe the cubic formula for factoring an arbitrary polynomial of degree 3 in $\mathbb{R}[x]$. This version of the formua is called the *Cardano-Tartaglia* formula after two 16th-century Italian mathematicians involved in its discovery. Consider the polynomial $f = x^3 + ax^2 + bx + c \in \mathbb{R}[x]$ (by dividing by the lead coefficient, if necessary, we have assumed without loss of generality that it is 1).

a) Show that the change of variables $x = y - \frac{1}{3}a$ changes f into a cubic polynomial that lacks a square term; that is, a polynomial of the form $g = f(y - \frac{1}{3}a) = y^3 + py + q = 0$.

b) Find explicit solutions u, v to the pair of simultaneous equations

$$v^3 - u^3 = q$$

$$uv = \frac{1}{3}p.$$

Hint: These equations reduce to a *quadratic* equation in u^3 or v^3.

c) Prove that the identity

$$(u-v)^3 + 3uv(u-v) + (v^3 - u^3) = 0$$

and use it to show that $y = u - v$ is a solution to the cubic equation $y^3 + py + q = 0$.

d) Let $D = q^2 + \frac{4p^3}{27}$. (This is called the *disciminant* of the conic.) Conclude that

$$y = \sqrt[3]{\frac{-q+\sqrt{D}}{2}} - \sqrt[3]{\frac{q+\sqrt{D}}{2}}$$

is a root for $g = 0$.

e) Use the formula to factor $p(x) = x^3 - 6x - 9$.

4) (# 26, Chapter 11) # Show that the rings

$$\mathbb{Z}_8, \ \mathbb{Z}_4 \times \mathbb{Z}_2, \text{ and } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

are all non-isomorphic, even though each of these rings has the same number of elements.

5) (# 11, Chapter 12) Consider the projective homomorphism $\pi_1 : R \times S \to R$ given by

$$\pi_1((r,s)) = r$$

for all $r \in R$, $s \in S$. What is the kernel of π_1? When do two elements of $R \times S$ get mapped to the same element of R? The set of pre-images of π_1 is naturally in one-to-one correspondence with what ring? Answer all of these questions with proof.

Extra Credit: Prove that $M_n(\mathbb{R})$ is simple. I will accept no written solutions; you must present your solution to me on the board in my office. You may receive some credit by demonstrating this fact for $M_3(\mathbb{R})$.