Math 412/512 Assignment 5

Due Thursday, November 19

1) (#5, Chapter 39) Let V be a vector space over the field $\mathbb{F}$. Suppose that $0 \neq r \in \mathbb{F}$. Define the function

$$\phi_r : V \to V$$

by $\phi_r(v) = rv$. Prove that ϕ_r is a bijection that preserves addition.

2) (#4, Chapter 41) Suppose that $\alpha \in \mathbb{C}$ is an algebraic number over $\mathbb{Q}$ and $r \in \mathbb{Q}$. Prove that $r\alpha$ and $\alpha + r$ are also algebraic over $\mathbb{Q}$.

Extra Credit: If

$$\mathbb{A} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q}\},$$

conclude that $\mathbb{A}$ is a field. No written solutions accepted, you must give your proof on the board in my office.

3) Give an example, with proof, of a noncommutative unital ring R and a maximal ideal I in R where R/I is not a field. You may use the fact that $M_n(\mathbb{R})$ is simple.

4) (#2, Chapter 42) Consider the polynomial

$$f(x) = 1 + x^2 + x^3 \in \mathbb{Z}_2(\alpha)[x]$$

where α is a root of $p(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$ over some extension field E. Recall that $\mathbb{Z}_2(\alpha)$ is a field with four elements.

a) Use the Root Theorem to show that $f(x)$ is irreducible.

b) By Kronecker's Theorem we can construct an extension field of $\mathbb{Z}_2(\alpha)$ so that $f(x)$ has a root β. How many elements does this field have? Be sure to prove your assertion!

5) Let $n, m \in \mathbb{N}$ and suppose $n \neq m$.

a) Find polynomials $p, q, r \in \mathbb{Z}[x]$ such that $p(\sqrt{5} + \sqrt{7}) = q(\sqrt{5}) = r(\sqrt{7}) = 0$. *Hint:* you'll need at least degree 4 for p.

b) Prove that $\mathbb{Q}[\sqrt{n}, \sqrt{m}] = \mathbb{Q}[\sqrt{n} + \sqrt{m}]$.

c) Compute the degree of $\mathbb{Q}[\sqrt{5}, \sqrt{7}]$, with a proof that your answer is correct.

d) Show that $\mathbb{Q}[\sqrt{7}]$ and $\mathbb{Q}[\sqrt{5}]$ are not isomorphic as fields. *Hint:* Prove that any isomorphism is the identity on $\mathbb{Q}$ and argue by contradiction.