Math 412/512 Assignment 6

Due Thursday, December 3

1) Tex up the problem that you were assigned via e-mail.

2) (#1, Chapter 46) Compute the Galois groups

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}) \text{ and } \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}(\sqrt{2})).$$

3) Let S_∞ denote the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$ that fix all but a finite number of elements, with function composition as the operation. So for example,

$$\phi(n) = \begin{cases} 1 & \text{if } n = 2 \\ 2 & \text{if } n = 1 \\ n & \text{if } n \geq 2 \end{cases}$$

determines an element $\phi \in S_\infty$ since only two elements of $\mathbb{N}$ are not fixed by ϕ, but

$$\gamma(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$$

does not determine an element of S_∞ since γ does not fix any elements of $\mathbb{N}$.

a) Show that S_∞ is a group.

b) Let $T \subset S_\infty$ be the subset of all bijections ϕ with the property that $\phi(1) = 1$. Is T is a subgroup of S_∞?

4) (#26, Chapter 20) Let G be a group and let $H = \{g \in G \mid g^2 = 1\}$. Show that H is a subgroup of G if G is abelian. Given an example WITH PROOF where this is false if G is not abelian.

5) Let $\langle G, \cdot \rangle$ be a group.

a) Suppose H and K are subgroups of $\langle G, \cdot \rangle$. Show that $H \cap K$ is also a subgroup of $\langle G, \cdot \rangle$.

b) Is it always the case that if H and K are subgroups of $\langle G, \cdot \rangle$ then $H \cup K$ is a subgroup of $\langle G, \cdot \rangle$? Prove or give a counterexample.

c) Let H be a subgroup of $\langle G, \cdot \rangle$ and for $x \in G$, let

$$xHx^{-1} = \{xax^{-1} : a \in H\}.$$

Define

$$K = \{x \in G : xHx^{-1} = H.\}$$

Show that $H \subseteq K$ and that K is a subgroup of $\langle G, \cdot \rangle$. K is called the *normalizer* of H in $\langle G, \cdot \rangle$, and is usually denoted by $N_G(H)$.

d) If $\langle G, \cdot \rangle$ is abelian, determine the normalizer of all subgroups H of $\langle G, \cdot \rangle$.

6) (#2, Chapter 21) Prove that every subgroup of a cyclic group is cyclic.