**Math 412/512 Final A**

**DIRECTIONS:** Take this final if you are shooting for an A in this course. The odd-numbered problems are definitions and theorems meant to aid you in the subsequent even-numbered problem. Use them wisely.

**1)** a) (6 points) Give the definition of a normal subgroup. Any equivalent condition will be accepted.

b) (7 points) State the First Isomorphism Theorem for groups.

**2)** a) (12 points) Let $G$ be a group, $n \in \mathbb{N}$, and suppose $H \leq G$ is the unique subgroup of order $n$ in $G$. Prove that $H$ is normal in $G$.

b) (13 points) Show that there can be no surjective homomorphism from $S_3$ to $\mathbb{Z}_3$.

**3)** a) (6 points) Define a cyclic group.

b) (6 points) State the definition of a ring isomorphism between two rings $R$ and $S$.

**4)** a) (10 points) Show that $2\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic as groups.

b) (15 points) Prove that $2\mathbb{Z}$ and $5\mathbb{Z}$ are not isomorphic as rings.

**5)** a) (6 points) Let $R$ be a ring. Define a zero divisor in $R$.

b) (5 points) Define a ring with unity.

**6)** a) (12 points) Let $S$ be a ring without unity and consider $R = S \times \mathbb{Z}$ as the set of ordered pairs $(s, n)$ with $s \in S$ and $n \in \mathbb{N}$. Define a nonstandard multiplication on $R$ by, for all $s_1, s_2 \in S$ and $n_1, n_2 \in \mathbb{Z}$,

$$(s_1, n_1) \times (s_2, n_2) = (s_1 s_2 + n_1 s_2 + n_2 s_1, n_1 n_2).$$

You may assume that, with the usual coordinate-wise addition and this multiplication, $R$ is a ring. Prove that $(0_S, 1)$ is the unity of $R$ and that $S$ is a subring of $R$.

b) (13 points) Let $T \in M_2(\mathbb{R})$ and suppose that $T \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T^2 = T$. T generates a unital, commutative subring $S$ of $M_2(\mathbb{R})$. Show that $T$ is a zero divisor in $S$.

**7)** a) (7 points) Let $R$ be a ring. Define what it means for $I \subseteq R$ to be an ideal of $R$.

b) (7 points) Let $V$ be a vector space over the field $F$. Define the dimension of $V$ over $F$.

**8)** Do ONE of the following two questions. If you do both, I will grade the one you do WORSE on.

a) (25 points) Let $S$ consist of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are discontinuous at zero, plus the zero function. Is $S$ a vector space over $\mathbb{R}$?

-OR-

b) (25 points) Show that if $R$ is a commutative ring with unity and $R$ has no proper, nontrivial ideals, then $R$ is a field.