Math 412/512 Final B

DIRECTIONS: If you choose this final, your grade in the course can be no higher than a B.

1) a) (5 points) State the definition of a group G.

b) (4 points) Define the order of a group G.

c) (4 points) Give the definition of an isomorphism between two groups G and H.

d) (5 points) Provide an example of two groups with finite order that are not isomorphic.

2) a) (5 points) State the 1-step subgroup test for determining whether a subset H of G is a subgroup of G.

b) (5 points) Define a normal subgroup of a group G. Any equivalent condition is acceptable.

c) (15 points) Show that if $H = \{T \in GL_2(\mathbb{R}) \mid \det(T) \in \mathbb{Q}\}$, then H is normal in $GL_2(\mathbb{R})$. NOTE: you should prove both that H is a subgroup and that H is normal.

3) a) (5 points) Define the index for a subgroup H of a group G.

b) (5 points) State Lagrange's Theorem

c) (5 points) Provide an example of a finite index inclusion $H \leq G$ where the order of G is infinite.

4) Define the following groups used in this course and give the group operation.

a) (4 points) $\mathbb{Z}$

b) (4 points) $GL_n(\mathbb{R})$, $n \in \mathbb{N}$.

c) (4 points) S_n, $n \in \mathbb{N}$, $n \geq 2$.

d) (4 points) D_n, $n \in \mathbb{N}$, $n \geq 3$

5) a) (5 points) Give the definition of a ring.

b) (3 points) State what it means for a ring to be commutative.

c) (3 points) State what it means for a ring to have a unity.

d) (5 points) Give an example of an infinite, non-commutative ring with unity.

6) a) (5 points) State the subring test for a subset S of a ring R.

b) (4 points) Define an ideal I of a ring R.

c) (15 points) Let $R = \mathbb{Z} \oplus \mathbb{Z}$ and $I = \{(3n, 5m) \mid n, m \in \mathbb{Z}\}$. Show I is an ideal of R.

7) a) (4 points) For a ring R, define what it means for $x \in R$ to be a unit.

b) (5 points) Define what it means for a ring R to be a field.

c) (4 points) Provide an example of a field.

8) a) (5 points) State the subspace test for a subset W of a vector space V.

b) (4 points) Define a basis for a vector space V.

c) (14 points) Prove that the following subset of $\mathbb{R}^3$ is a subspace and find a basis.

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid 2z + 7y + 11x = 0\}.$$