Math 412/512 Final

Directions: The odd numbered problems are definitions and theorems meant to aid you in the subsequent even-numbered problem, with the possible exception of 5b). Use them wisely.

1) a) Define what it means for a subgroup H of a group G to be normal.

b) State Lagrange's Theorem for a finite group.

2) a) Let G be a group and let $H \leq G$, $K \leq H$. Suppose $[G : K] < \infty$. Prove that $[G : K] = [G : H] \cdot [H : K]$.

b) Let G be a group and suppose $[G : H] < \infty$. Show that $H \triangleleft G$ if and only if $gHg^{-1} \subseteq H$ for all $g \in G$.

3) a) Let R be a ring. Define what it means for $I \subseteq R$ to be an ideal of R.

b) Give the definition of a field.

4) a) Provide an example, without proof, of a noncommutative ring with identity R that contains no proper, nontrivial ideals.

b) Let R be any commutative ring with identity and suppose R contains no proper, nontrivial ideals. Prove that R must be a field.

5) a) Define the symmetric group S_n $(n \geq 2)$ and the group D_n $(n \geq 3)$.

b) State the First Isomorphism Theorem for groups.

6) a) Prove that $S_3 \times D_4$ is not isomorphic to S_4.

b) Show that $S_3 \times Z_4$ is not isomorphic to S_4.

7) a) State the Fundamental Theorem of Field Theory.

b) Let F be a field and let $p \in F[x]$. Define the splitting field of p.

8) Do ONE of the following two problems.

a) Let $R = \mathbb{Z} \times \mathbb{Z}$. Describe all maximal ideals in R and determine all ring homomorphisms from R to $\mathbb{Z}$.

-OR-

b) Consider $p(x) = x^4 + x^2 + 1 \in \mathbb{Q}[x]$. Find the splitting field of p.