Math 412/512 Final

DIRECTIONS: The odd-numbered problems are definitions and theorems meant to aid you in the subsequent even-numbered problem. Use them wisely.

1) Let R be a ring.

a) Give the subring test for a subset S of R.

b) State the Ring Isomorphism Theorem.

2) Let $R = M_2(\mathbb{Z})$ Let

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in 2\mathbb{Z} \right\}.$$

a) Prove that S is a subring of R.

b) What is also true, but that you do not have to prove, is that S is an ideal in R. Identify R/S isomorphically with $M_2(\mathbb{Z}_2)$; be sure to prove that your map is, indeed, an isomorphism.

3) Let $\mathbb{F}$ be a field.

a) Define what it means for $p(x) \in \mathbb{F}[x]$ to be irreducible.

b) State Kronecker's Theorem.

4) Let $p(x) = x^5 + 12x^3 - 18x^2 + 9x + 3$.

a) Prove that $p(x)$ is irreducible over $\mathbb{Q}$.

b) Let $\alpha \in \mathbb{C}$ be a root of $p(x)$. Find a basis for $\mathbb{Q}(\alpha)$ and compute the degree of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$.

5) a) Let R be a ring with unity. Define $R^\times$, the group of units of R.

b) Let G be a group. Define the order of an element $g \in G$.

6) a) Let $G = \mathbb{Z}_8^\times$ with the operation of multiplication modulo 8. Prove that G is not cyclic.

b) Let $\phi : G \to H$ be a surjective group homomorphism and suppose that $\text{ord}(g) = \text{ord}(\phi(g))$ for all $g \in G$. Prove that ϕ is an isomorphism.

7) a) Define a maximal ideal of a ring R.

b) Define a principal ideal domain.

8) Do ONE of the following two questions. If you do both, I will grade the one you do WORSE on.

a) Construct a field $\mathbb{F}$ with 9 elements. NOTE: $\mathbb{Z}_9$ is NOT a field!

-OR-

b) Let R be an integral domain and let $a \in R$ be a nonzero, noninvertible element of R. Prove that $\langle a, x \rangle \subset R[x]$ is not a principal ideal domain.