Math 412/512 Midterm

IN-CLASS PORTION

1) a) (5 points) Let G be a set with a binary operation "$*$". Under what conditions is $\langle G, * \rangle$ a group? i.e., Give the definition of a group.

b) (4 points) Give an example of an infinite, nonabelian group.

c) (4 points) Give an example of a finite, abelian group.

2) Suppose G is a group and $H \subset G$.

a) (5 points) What conditions must H satisfy to be a subgroup of G? You may either give the definition of a subgroup or a test which determines H is a subgroup.

b) (4 points) If $G = \mathbb{R} \setminus \{0\}$ with the operation of multiplication, exhibit a nontrivial (i.e. neither G nor $\{e_G\}$) subgroup of G.

3) a) (5 points) For $n \in \mathbb{N}$, define the symmetric group S_n.

b) (4 points) For an element $\sigma \in S_n$, define what it means for σ to be even.

c) (3 points) Give an example of an even permutation in S_4.

d) (5 points) State Cayley's Theorem for a finite group G.

4) Let $\langle G, * \rangle$ and $\langle H, \cdot \rangle$ be groups.

a) (4 points) Give the definition of a homomorphism from $\langle G, * \rangle$ to $\langle H, \cdot \rangle$.

b) (3 points) What additional condition(s) must a homomorphism satisfy in order to be an isomorphism of groups?

c) (4 points) Define the order of a group.

d) (5 points) Provide an example of two groups with the same order that are not isomorphic.

5) Let $H \leq G$.

a) (5 points) Define the left cosets of H in G.

b) (5 points) State Lagrange's Theorem for a finite group G.

c) (5 points) Define what it means for H to be a normal subgroup of G.

d) (5 points) Give an example of a nontrivial (i.e., neither S_4 nor the identity element) normal subgroup of S_4.

Due Friday, March 11

1) Recall from class that

$$U(n) = \{x \in \mathbb{Z}_n : \text{there exists } y \in \mathbb{Z}_n \text{ with } xy \equiv 1(\text{modulo n})\}.$$

We know that $|U(n)| = \phi(n)$.

a) Calculate $|U(15)|$.

b) Show that $U(15)$ is not cyclic. *Warning:* quoting any general theorem from which this result follows as a trivial consequence will necessitate that you also provide a proof of said theorem.

2) Let G be a group, $H \leq G$, and suppose $[G : H] = 2$. Prove that $H \triangleleft G$.

3) Recall that S_n denotes the group of all bijections on a set with n elements, with the operation of function composition. Prove that for all natural numbers n and m, S_{n+m} has a subgroup isomorphic to $S_n \times S_m$.

4) The notation $M_2(\mathbb{R})$ refers to the group of all 2×2 matrices with real entries, the group operation being matrix addition. Let

$$\mathcal{S} = \{T \in M_2(\mathbb{R}) : T^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

Is $\mathcal{S}$ a subgroup of $M_2(\mathbb{R})$? Prove your assertion.

5) Let G and H be groups and suppose $\phi : G \to H$ is a map satisfying $\phi(e_G) = e_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$. Does it then follow that ϕ is a homomorphism? Prove or give a counterexample.