Math 412/512 Midterm Part 1

1) a) (5 points) For a ring R, define what it means for R to be commutative and to have unity.

b) (4 points) Give an example of a noncommutative ring with unity.

c) (4 points) Give an example of a commutative ring without unity.

2) Let R be a ring.

a) (5 points) Define an ideal I of R.

b) (4 points) Define a maximal ideal of R.

c) (3 points) Give an example of a maximal ideal in $\mathbb{Z}$.

3) Let R be a ring.

a) (5 points) Define a zero divisor of R.

b) (4 points) Define an integral domain.

c) (3 points) Give an example of a finite integral domain.

4) Let R and S be rings.

a) (4 points) Give the definition of a ring homomorphism from R to S.

b) (2 points) What additional condition(s) must a homomorphism satisfy in order to be an isomorphism of rings?

c) (4 points) State the fundamental isomorphism theorem for rings.

d) (5 points) Provide examples of two infinite rings that are not isomorphic.

5) a) (6 points) State the Unique Factorization Theorem in $\mathbb{Z}$.

b) (4 points) Define an irreducible polynomial in $\mathbb{Z}[x]$.

c) (6 points) State Eisenstein's test for polynomials in $\mathbb{Z}[x]$.

6) (12 points) Let $R = M_2(\mathbb{R})$ and let

$$\mathcal{S} = \left\{ A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_2(\mathbb{R}) |\ a_{1,2} = 0 \right\}$$

Prove that S is a subring of R.

Math 412/512 Midterm Part 2

1) Let R be a commutative ring. Prove that if $xy \in R$ is a zero divisor, then either x or y is a zero divisor.

2) Prove that $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic as rings.

3) Prove that every ideal in $\mathbb{Z} \times \mathbb{Z}$ is principal.