Application: (determinants) Let

$$A = (A_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{R})$$

(can substitute your favorite field for $\mathbb{R}$)

We can now fully define the **determinant** of A to be

$$\det(A) = \sum_{\sigma \in S_n} \left(\varepsilon(\sigma) \cdot A_{1,\sigma(1)} \cdot A_{2,\sigma(2)} \cdot \ldots \cdot A_{n,\sigma(n)} \right)$$

$$= \sum_{\sigma \in S_n} \left(\varepsilon(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)} \right)$$

Check for $n=2$

2 permutations, e and $\sigma = (12)$

$$\det(A) = \underbrace{\varepsilon(e)}_{1} A_{1,e(1)} \cdot A_{2,e(2)}$$

$$+ \underbrace{\varepsilon(\sigma)}_{-1} A_{1,\sigma(1)} \cdot A_{2,\sigma(2)}$$

$$\det(A) = A_{1,1} A_{2,2} - A_{1,2} A_{2,1}$$ ✓