Example 3:

The triangle inequality for real numbers:

if $x, y, z \in \mathbb{R}$, then

$$|x-z| \leq |x-y| + |y-z|.$$

Proof: By cases. Let $a = x-y$, $b = y-z$. We reduce to proving

$$\begin{aligned} |a+b| &= |x-y+y-z| \\ &= |x-z| \\ &\leq |x-y| + |y-z| \\ &= |a| + |b|. \end{aligned}$$

Case 1: $a \geq 0, b \geq 0$

Then $|a| = a$, $|b| = b$

$|a+b| = a+b$, and

so we have equality:

$$|a+b| = a+b = |a| + |b|$$

Case 2: $a < 0, b < 0$.

Then $|a| = -a$, $|b| = -b$

$|a+b| = -(a+b) = -a-b$

and we again have equality:

$$|a+b| = -a-b = |a| + |b|$$

Case 3: $a \geq 0,\ b < 0$

Then $|a| = a$, $|b| = -b$.

If

$a+b \geq 0$, then

$$|a+b| = a+b \leq a+(-b) = |a|+|b|$$

If $a+b<0$, then

$$\begin{aligned}|a+b| &= -(a+b) = (-a)+(-b)\\ &\leq a+(-b)\\ &= |a|+|b|\end{aligned}$$

Case 4: $a < 0,\ b \geq 0.$

Then $|a| = -a,\ |b| = b.$

If $a+b \geq 0,$

$$|a+b| = a+b \leq (-a)+b = |a|+|b|.$$

If $a+b < 0$

$$\begin{aligned} |a+b| &= -(a+b) = (-a)+(-b) \\ &\leq (-a)+b \\ &= |a|+|b| \end{aligned}$$

$\square$

Remark: In cases like 3) and 4), where the similarities are apparent, you are allowed to quote something to the effect "(case 4) is similar to (case 3), and so the proof is omitted."

But you had better be right!

Standard Mathematical Symbols

"$\mathbb{N}$" = the "natural" numbers

$1, 2, 3, 4, 5, 6, \ldots$

"$\mathbb{Z}$" = the integers

= natural numbers and their negatives, along with zero

"$\mathbb{Q}$" = the rational numbers

$$= \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

"$\mathbb{R}$" = the real numbers

= the "completion" of $\mathbb{Q}$

"$\mathbb{C}$" = the complex numbers

$= \{ a + bi \mid a, b \in \mathbb{R} \}$

where $i^2 = -1$.