Q: Can you construct a field of any (finite) cardinality?

A: No! The cardinality of any finite field must be p^n for some prime p and some $n \in \mathbb{N}$.

Our example works since $4 = 2^2$, $p = n = 2$.

Why there is no field with 6 elements:

Call the potential field L. If $|L|=6$, L has two operations "+" and "$\cdot$" such that, if 0 is the additive identity for L,

$(L,+)$ is a commutative group and

$(L\setminus\{0\},\cdot)$ is a commutative group.

Since $|L|=6$, $(L,+)$ is isomorphic to $\mathbb{Z}_6$ as a group. Similarly,

$(L\setminus\{0\},\cdot)$ is isomorphic to $\mathbb{Z}_5$.

Since $(L, +)$ is isomorphic to $\mathbb{Z}_6$,

note that in $\mathbb{Z}_6$,

$$[2]+[2]+[2]=[6]=[0]$$

$$[3]+[3]=[6]=[0]$$

$$[4]+[4]+[4]=[12]=[0]$$

$$[1]+[1]+[1]+[1]+[1]+[1]=[6]=[0]$$

$$[5]+[5]+[5]+[5]+[5]+[5]=[30]=[0]$$

(minimal number of additions)

Since $(L\setminus\{0\}, \cdot)$ is isomorphic

to $\mathbb{Z}_5$, there is an element

$a \in L$, $a \neq 0$, with

$$\{a, a^2, a^3, a^4, a^5 = 1\} = (L\setminus\{0\}, \cdot)$$

(a is just the image of $[1]_5$ under

the isomorphism)

Then $L = \{0, 1, a, a^2, a^3, a^4\}$.

Consider adding 1 to itself.

By no more than five iterations,

you will return to zero, since

$(L, +)$ is isomorphic to $\mathbb{Z}_6$.

$$\underbrace{1+1+\dots+1}_{\text{no more than 6 ones}} = 0$$

Suppose $1+1=0$.

Then

$$a+a = a(1+1) = a\cdot 0 = 0.$$

Similarly, $a^2+a^2=0$, $a^3+a^3=0$, $a^4+a^4=0$.

We know that $1+1 \neq 0$ because in $\mathbb{Z}_6$, you can't add [5] to itself twice and get [0].

If $1+1+1=0$, then

$a+a+a = a(1+1+1) = a \cdot 0 = 0$

and also $a^2+a^2+a^2 = a^3+a^3+a^3 = a^4+a^4+a^4 = 0$.

This can't happen for the same reason as for $1+1=0$.

There is no nonzero element x in $\mathbb{Z}_6$ with $x+x+x+x=[0]$ or

$x+x+x+x+x=[0]$

By process of elimination,

$1+1+1+1+1+1=0$

(minimal number of additions)

But since $(L,+)$ is isomorphic to $\mathbb{Z}_6$ as a group, $\exists\ y \in L$,

$y+y=0$, $y \neq 0$. But

Since $y \neq 0$, y is invertible!

Then

$$0 = y+y = y(1+1).$$

Multiply both sides by y^{-1} to get

$$0 = y^{-1}\cdot 0 = \underbrace{y^{-1}\cdot y}_{1}\cdot(1+1)$$

so $0 = 1+1$, contradiction!