Another Example In S_6, let

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 5 & 2 & 1 \end{bmatrix}$$

$$g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{bmatrix}$$

Compute $f \circ g$ and $g \circ f$

$$f \circ g : \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 5 & 2 & 1 \end{bmatrix}$$

$$f \circ g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 & 4 & 3 \end{bmatrix}$$

$$gof : \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{bmatrix}$$

$$gof = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 1 & 5 & 6 \end{bmatrix}$$

Note: $fog \neq gof$

Definition: (cycle, disjoint cycles)

A **cycle** in S_n is a permutation f such that we can write

$$\{1, 2, \ldots, n\} = T_1 \cup T_2$$

with $T_1 \cap T_2 = \emptyset$

where $\forall\ k \in T_2$, $f(k) = k$

and $\forall\ a, b \in T_1$, $\exists\ m \in \mathbb{N}$

with $f^{(m)}(a) = b$.

Here, $f^{(m)} = \underbrace{f \circ f \circ f \circ \cdots \circ f}_{m \text{ times}}$

and $\forall\ a \in T$, $f(a) \neq a$.

Two cycles f and g are said to be disjoint if whenever $f(k) = k$ whenever $g(k) \neq k$ and $g(k) = k$ whenever $f(k) \neq k$.

Example 2: (cycles) In S_5,

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{bmatrix}$$

is a cycle with

$$T_1 = \{1, 3, 4\}$$

and $T_2 = \{2, 5\}$.

$f(1) = 4$

$f(f(1)) = f(4) = 3$

$f(3) = 1$

$f(f(3)) = f(1) = 4$ etc.

If

$$g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{bmatrix}$$

then for g,

$$T_1 = \{2, 5\}$$

$$T_2 = \{1, 3, 4\}$$

So f and g are disjoint.

Not a cycle:

$$h = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{bmatrix}$$

h is not a cycle since $h(3) \neq 3$ and $h(1) \neq 1$, but there is no power of h with $h(1) = 3$.

Cycle Notation

If $f \in S_n$ is a cycle,

write

$$T_1 = \{x_1, x_2, \dots, x_m\}$$

where $m = |T_1|$ and

$$x_2 = f(x_1),\ x_3 = f(x_2),\ \dots,\ x_1 = f(x_m)$$

Then we use cycle notation to describe f:

$$\boxed{f = (x_1\, x_2\, \dots\, x_m)}$$

Example 3: (expressing permutations as cycles)

Start with a cycle

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 2 & 1 & 6 \end{bmatrix}$$

In cycle notation,

$$f = (1\,4\,2\,5).$$

Read this from left to right as

"1 goes to 4, 4 goes to 2, 2 goes to 5, 5 goes to 1".

Note that we suppress any numbers that f does not move.

Q: What if f fixes all the numbers?

A: Then f is the identity permutation, denoted by e.

Composition of Cycles

If $f, g \in S_n$ are cycles,

$$f = (x_1\ x_2 \ldots x_m)$$

$$g = (y_1\ y_2 \ldots y_k)$$

for $m, k \leq n$, then the composition $f \circ g$ is given as

$$\boxed{f \circ g = (x_1\ x_2 \cdots x_m)(y_1\ y_2 - y_k)}$$

Note: if f and g are disjoint, this notation is clear, and $f \circ g = g \circ f$. What if f and g are not disjoint?

Then the cycle to the farthest right is the first permutation to be applied, then proceed from right to left (function composition)

Example 4: (composition of cycles) In S_3,

let $f = (123)$, $g = (12)$.

Compute $f \circ g$ in two ways:

1) By converting to 2-line notation and multiplying

2) By multiplying cycles.

1) $f = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

$g = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$

$$f \circ g : \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

In cycle notation,

$$f \circ g = (1\ 3)$$

2) $(1\,2\,3)(1\,2) = (1\,3)$ ✓

But: The product of cycles need not be a cycle!

In S_4, let $f = (1\ 2\ 3\ 4)$.

$$f \circ f = (1\ 2\ 3\ 4)(1\ 2\ 3\ 4)$$

$$f \circ f = (1\ 3)(2\ 4)$$ not a cycle

One last example: In S_7,

let $f = (1\ 2\ 5)(6\ 7)$

$g = (2\ 3\ 7\ 4)$

$h = (2\ 4)(5\ 7)(3\ 1)$

Compute!

$f \circ g \circ h =$

$$\Big((1\ 2\ 5)(6\ 7)\Big)\Big((2\ 3\ 7\ 4)\Big)\Big((2\ 4)(5\ 7)(3\ 1)\Big)$$

$$= (1\ 6\ 7)(2\ 5\ 4\ 3)\ \checkmark$$