

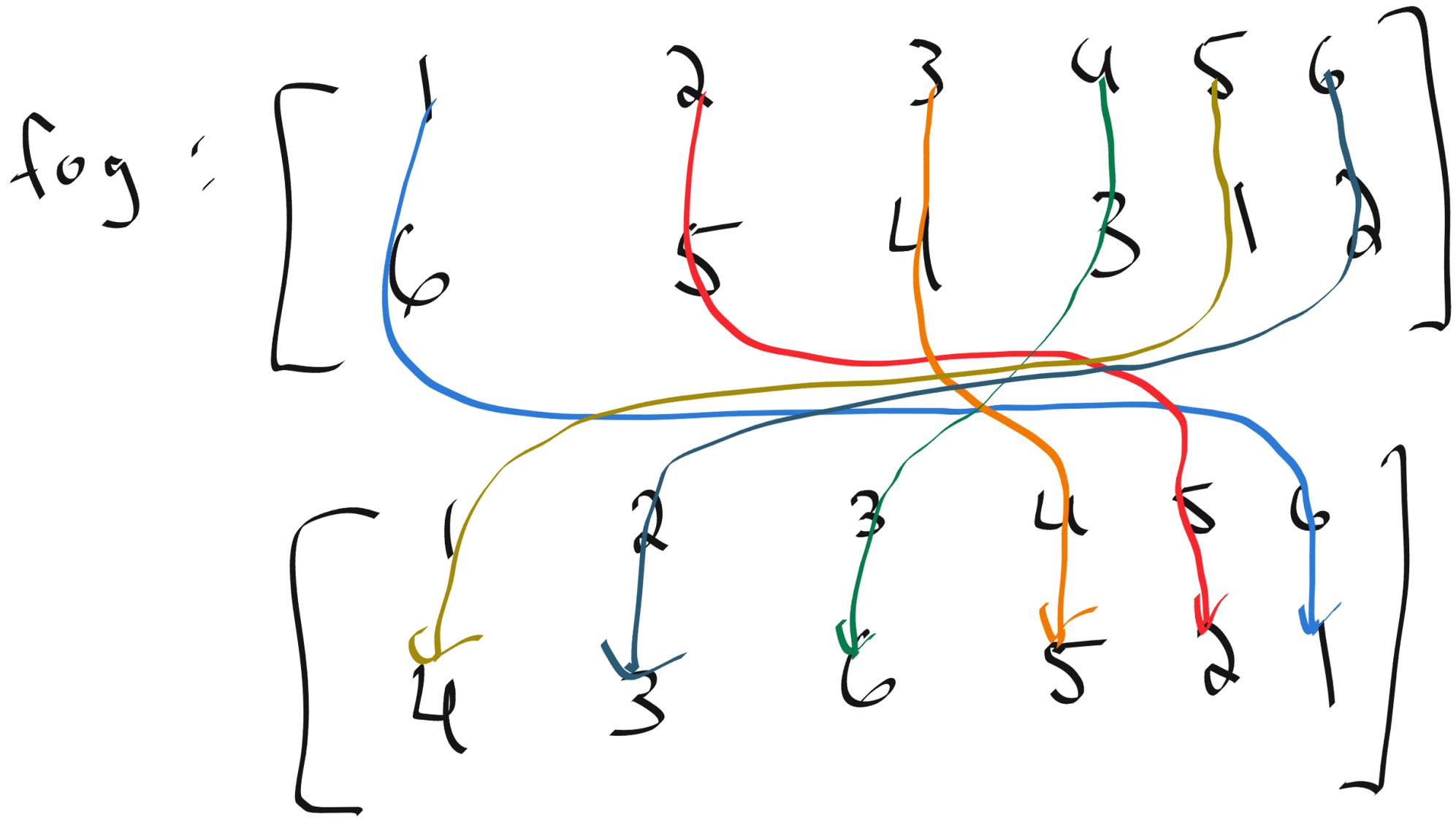
Another Example

In S_6 , let

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 5 & 2 & 1 \end{bmatrix}$$

$$g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{bmatrix}$$

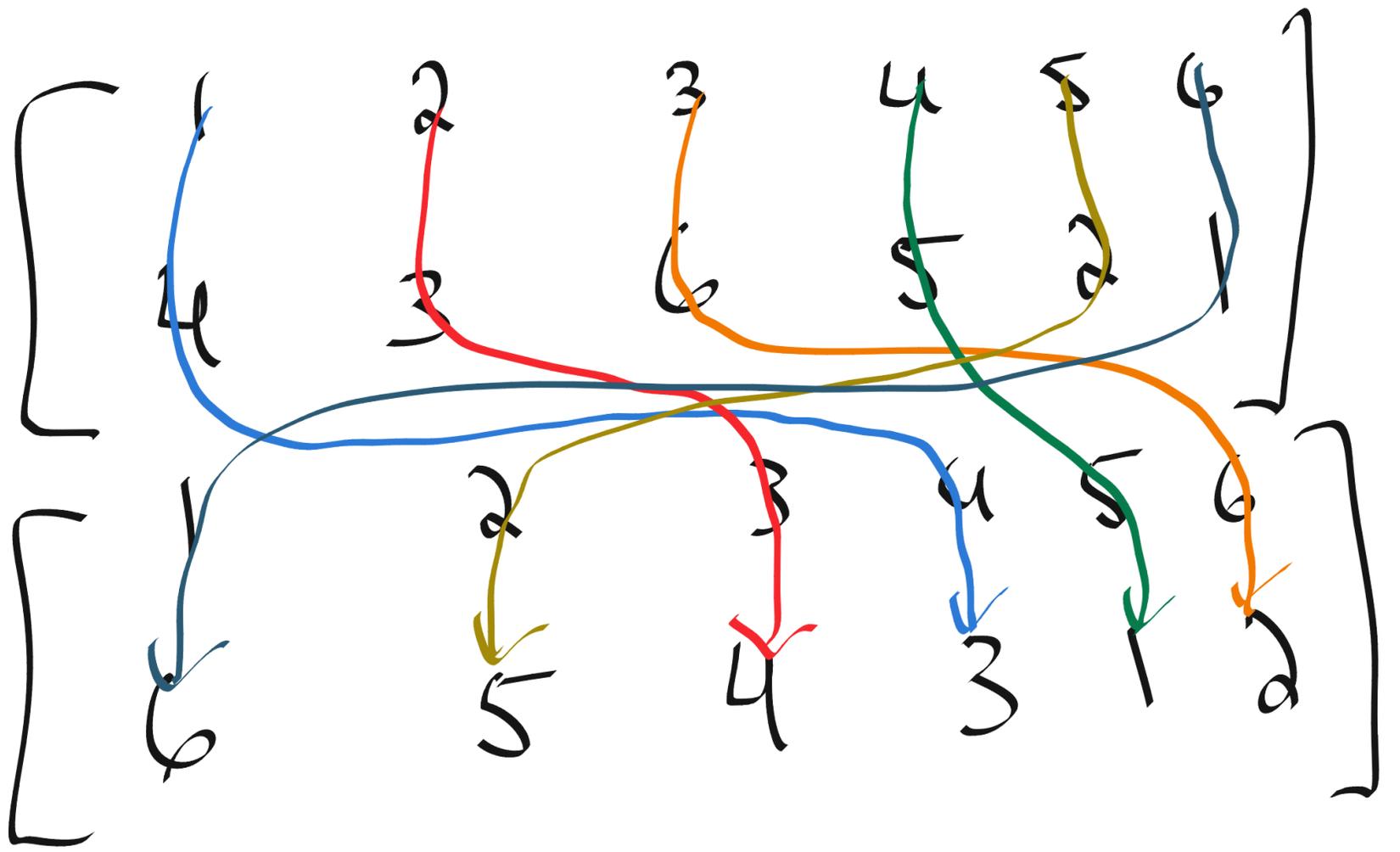
Compute $f \circ g$ and $g \circ f$



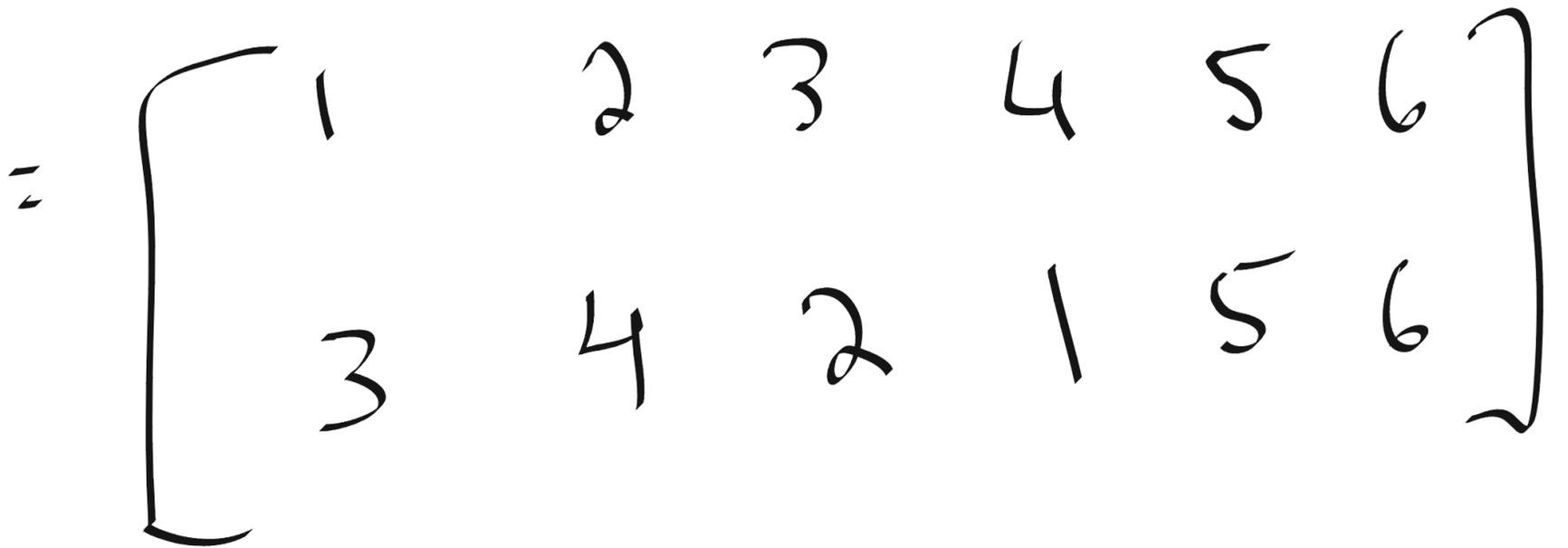
$f \circ g =$

1	2	3	4	5	6
4	5	6	3	2	1

$g \circ f$:



$g \circ f$ =



Note: $f \circ g \neq g \circ f$

Definition: (cycle, disjoint cycles)

A cycle in S_n is

a permutation f such

that we can write

$$\{1, 2, \dots, n\} = T_1 \cup T_2$$

$$\text{with } T_1 \cap T_2 = \emptyset$$

where $\forall k \in T_2, f(k) = k$

and $\forall a, b \in T_1, \exists m \in \mathbb{N}$

$$\text{with } f^{(m)}(a) = b.$$

Here, $f^{(m)} = \underbrace{f \circ f \circ f \circ \dots \circ f}_{m \text{ times}}$

and $\forall a \in T, f(a) \neq a$.

Two cycles f and g

are said to be **disjoint**

if whenever $f(k) = k$ whenever

$g(k) \neq k$ and $g(k) = k$ whenever

$f(k) \neq k$.

Example 2: (cycles) In S_5 ,

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{bmatrix}$$

is a cycle with

$$T_1 = \{1, 3, 4\}$$

and $T_2 = \{2, 5\}$.

$$f(1) = 4$$

$$f(f(1)) = f(4) = 3$$

$$f(3) = 1$$

$$f(f(3)) = f(1) = 4 \quad \text{etc}$$

If

$$g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{bmatrix}$$

then for g ,

$$T_1 = \{2, 5\}$$

$$T_2 = \{1, 3, 4\}$$

So f and g are disjoint.

Not a cycle:

$$h = \begin{bmatrix} & 1 & 2 & 3 & 4 & 5 \\ 2 & & 1 & 4 & 5 & 3 \end{bmatrix}$$

h is not a cycle since

$$h(3) \neq 3 \text{ and } h(1) \neq 1,$$

but there is no power of

$$h \text{ with } h(1) = 3.$$

Cycle Notation

If $f \in S_n$ is a cycle,

write

$$T_1 = \{x_1, x_2, \dots, x_m\}$$

where $m = |T_1|$ and

$$x_2 = f(x_1), x_3 = f(x_2), \dots, x_1 = f(x_m)$$

Then we use **cycle notation**
to describe f :

$$f = (x_1 x_2 \dots x_m)$$

Example 3: (expressing permutations as cycles)

Start with a cycle

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 2 & 1 & 6 \end{bmatrix}$$

In cycle notation,

$$f = (1\ 4\ 2\ 5)$$

Read this from left to right as

"1 goes to 4, 4 goes to 2,

2 goes to 5, 5 goes to 1".

Note that we suppress any numbers that f does not move.

Q: What if f fixes all the numbers?

A: Then f is the identity permutation, denoted by e .

Composition of Cycles

If $f, g \in S_n$ are cycles,

$$f = (x_1 x_2 \dots x_m)$$

$$g = (y_1 y_2 \dots y_k)$$

for $m, k \leq n$, then the

composition $f \circ g$ is given as

$$f \circ g = (x_1 x_2 \dots x_m) (y_1 y_2 \dots y_k)$$

Note: if f and g are disjoint, this notation is clear, and $f \circ g = g \circ f$. What if f and g are **not** disjoint?

Then the cycle to the farthest right is the first permutation to be applied, then proceed from right to left (**function composition**)

Example 4: (composition of cycles) In S_3 ,

let $f = (123)$, $g = (12)$.

Compute $f \circ g$ in two ways:

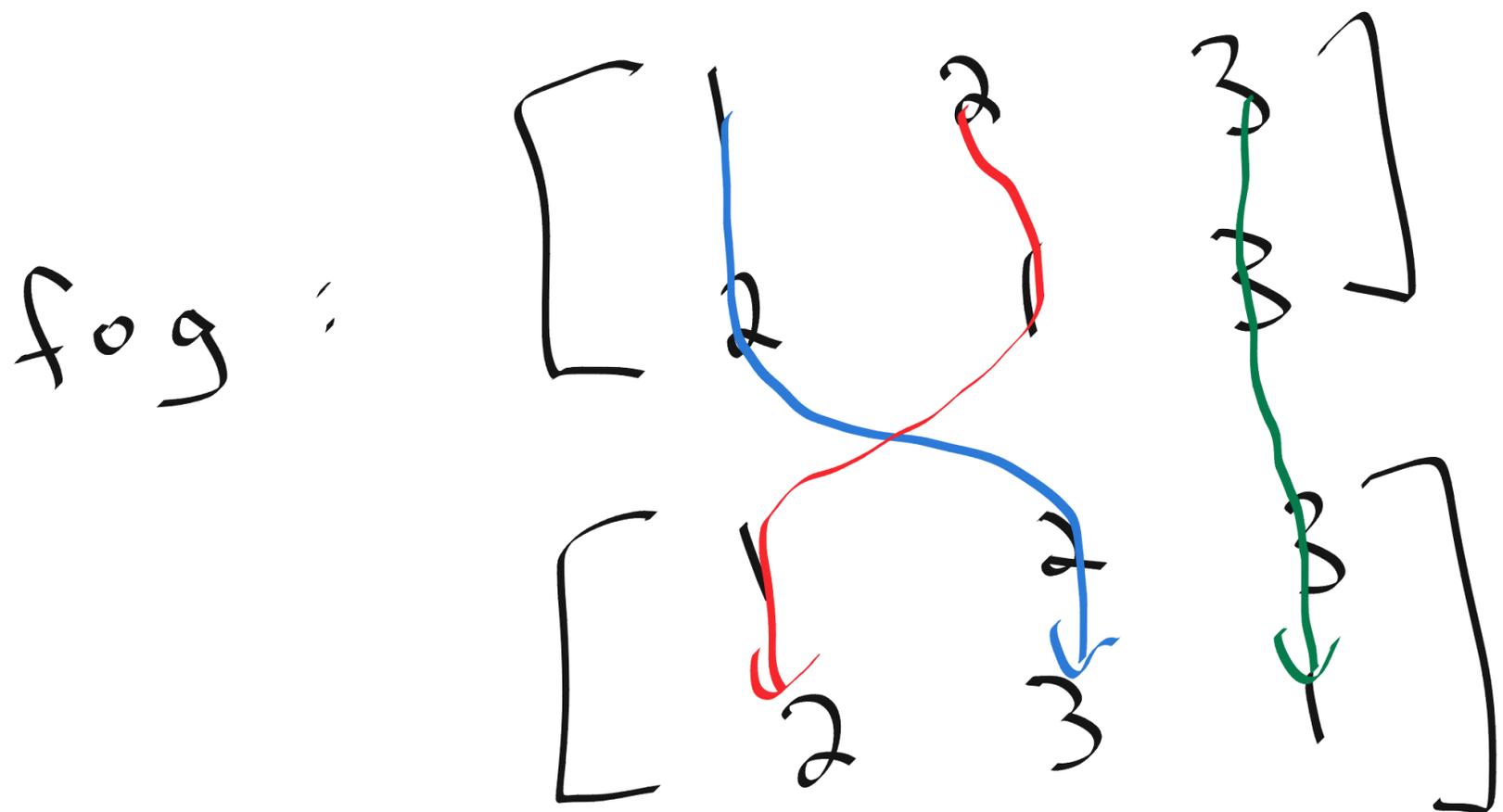
1) By converting to 2-line notation and multiplying

2) By multiplying cycles.

$$f = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$g = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

fog :



$$f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

In cycle notation,

$$f \circ g = (1 \ 3)$$

$$2) \quad (123)(12) = (13) \quad \checkmark$$

But: The product of cycles need
not be a cycle!

In S_4 , let $f = (1234)$.

$$f \circ f = (1234)(1234)$$

$$f \circ f = (13)(24) \quad \text{not a cycle}$$

One last example: In S_7 ,

$$\text{let } f = (1\ 2\ 5)(6\ 7)$$

$$g = (2\ 3\ 7\ 4)$$

$$h = (2\ 4)(5\ 7)(3\ 1)$$

Compute!

$$f \circ g \circ h =$$

$$\left((1\ 2\ 5)(6\ 7) \right) \left((2\ 3\ 7\ 4) \right) \left((2\ 4)(5\ 7)(3\ 1) \right)$$

$$= (1\ 6\ 7)(2\ 5\ 4\ 3) \quad \checkmark$$