Groups

(Section 1.10)

Examples so far: symmetries of polygons/polyhedra, permutations, $\mathbb{Z}$ (under addition), and $\mathbb{Z}_n$ (under addition) are all examples of **groups**.

Definition: (group) A **group** is a set G endowed with a binary operation "$\cdot$" : $G \times G \to G$ such that

Closed under "$\cdot$"

1) "$\cdot$" is associative.

2) $\exists$ an element $e \in G$, called the **identity**, such that $\forall\ g \in G$,

$$e \cdot g = g \cdot e = g.$$

3) $\forall\ g \in G$, $\exists\ h \in G$, called the **inverse** of g, such that

$$g \cdot h = h \cdot g = e$$

Example 1: ($GL_2(\mathbb{R})$) Denote by $GL_2(\mathbb{R})$, the general linear group of 2×2 matrices with real entries, such that a 2×2 matrix A is in $GL_2(\mathbb{R})$ when A^{-1} exists. So,

$$GL_2(\mathbb{R}) = \{A \text{ } 2\times 2 \text{ matrices} \mid A^{-1} \text{ exists}\}$$

Then we will show $GL_2(\mathbb{R})$ is a group!

Associativity: the product "$\cdot$" on $GL_2(\mathbb{R})$ is ordinary matrix multiplication.

Matrix multiplication is just function composition of the associated linear transformations on $\mathbb{R}^2$, and function composition is associative.

identity The identity of $GL_2(\mathbb{R})$ under "$\cdot$" is the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Check: Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$.

Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \checkmark$$

Inverses

We know that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is invertible, then

$$\det(A) = ad - bc \neq 0.$$

I claim that

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

is the inverse of A.

Check:

$$A \cdot A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

$$= \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & \underbrace{\cancel{ab-ba}}_{0} \\ \underbrace{\cancel{cd-dc}}_{0} & ad-bc \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now check $A^{-1} \cdot A$.

$$A^{-1} \cdot A = \left(\frac{1}{ad-bc}\right) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & \cancelto{0}{bd-bd} \\ \cancelto{0}{ac-ac} & ad-bc \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $GL_2(\mathbb{R})$ is a group!

Not so fast!

We need to check that "$\cdot$" is a binary operation. This means:

if $A, B \in GL_2(\mathbb{R})$, is

$$A \cdot B \in GL_2(\mathbb{R})?$$

Since $A \in GL_2(\mathbb{R})$, A^{-1} exists.

Similarly, $B \in GL_2(\mathbb{R})$, so B^{-1} exists.

We know $A^{-1}, B^{-1} \in GL_2(\mathbb{R})$.

We need to show that AB is invertible.

Claim: $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

Check: $(A \cdot B)(B^{-1} \cdot A^{-1})$

$= A \cdot (B \cdot B^{-1}) A^{-1}$ (associativity)

$= A \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1}$

$= A \cdot A^{-1}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$(B^{-1} \cdot A^{-1})(AB)$$

$$= B^{-1}(A^{-1} \cdot A) \cdot B$$ (associativity again)

$$= B^{-1} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot B$$

$$= B^{-1} \cdot B$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ✓

Now we know $GL_2(\mathbb{R})$ is a group!