Definition: (order of a group)

The **order** of a group G is simply the cardinality of G when G is finite. When G is infinite, we usually make no distinction between cardinalities and refer to all such groups as **infinite order**.

Definition: (abelian group) If G is a group with operation "$\cdot$" and "$\cdot$" is commutative, i.e.,

$$x \cdot y = y \cdot x \quad \forall\, y, x \in G,$$

we say G is an abelian group (after the mathematician Niels Abel).

Proposition: (groups of order less than 7)

Up to isomorphism, there is/are

1) One group of order 1.
2) One group of order 2.
3) One group of order 3.
4) Two groups of order 4 (both abelian)
5) One group of order 5.
6) Two groups of order 6, only one of which is abelian.

proof: 1) Suppose G, H are two groups, $|G| = |H| = 1$.

Then $G = \{e_G\}$, $H = \{e_H\}$

and $\varphi: G \to H$,

$\varphi(e_G) = e_H$ is

an isomorphism.

(e_G = identity of G,

e_H = identity of H,

must exist in any group)

2) Suppose $|G| = 2$. Then

$G = \{e_G, x\}$. If "$\cdot$"

denotes the binary operation,

then either

$$x \cdot x = e_G \quad \text{or}$$

$$x \cdot x = x.$$

But if $x \cdot x = x$, then

by cancellation,

$$x \cdot x = x \cdot e_G, \quad \text{so}$$

$$x = e_G \Rightarrow |G| = 1,$$

contradiction

Therefore, $x \cdot x = e_G$.

Define

$$\varphi : \mathbb{Z}_2 \to G$$

$$\varphi([0]) = e_G$$

$$\varphi([1]) = x.$$

Then φ is an isomorphism.

(check)

3) Suppose $|G| = 3$.

Then $G = \{e_G, x, y\}$.

Consider $x \cdot y$.

If $x \cdot y = y$ or $x \cdot y = x$,

then by cancellation, we have

$x = e_G$ or $y = e_G$, respectively,

contradiction.

Therefore, $x \cdot y = e_G$, and

So $y = x^{-1}$.

Then we must have

$x \cdot x = y$ since

if $x \cdot x = x$, then by

cancellation, $x = e_G$ and

if $x \cdot x = e_G$, then

$x = x^{-1} = y$. Both are

contradictions.

Similarly, $y \cdot y = x$.

Define $\varphi: \mathbb{Z}_3 \to G$

$$\varphi([0]) = e_G$$

$$\varphi([1]) = x$$

$$\varphi([2]) = y = x^{-1}.$$

Then φ is an isomorphism

(check)

Note that if instead, we had $\varphi([1]) = y$, $\varphi([2]) = x$, this is still an isomorphism!

4) The groups are $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the group structure

$$([x],[y]) \cdot ([z],[w]) = ([xz],[yw])$$

5) $\mathbb{Z}_5$ is the only group of order 5.

6) $\mathbb{Z}_6$ and S_3 are the only groups of order 6.

$\square$