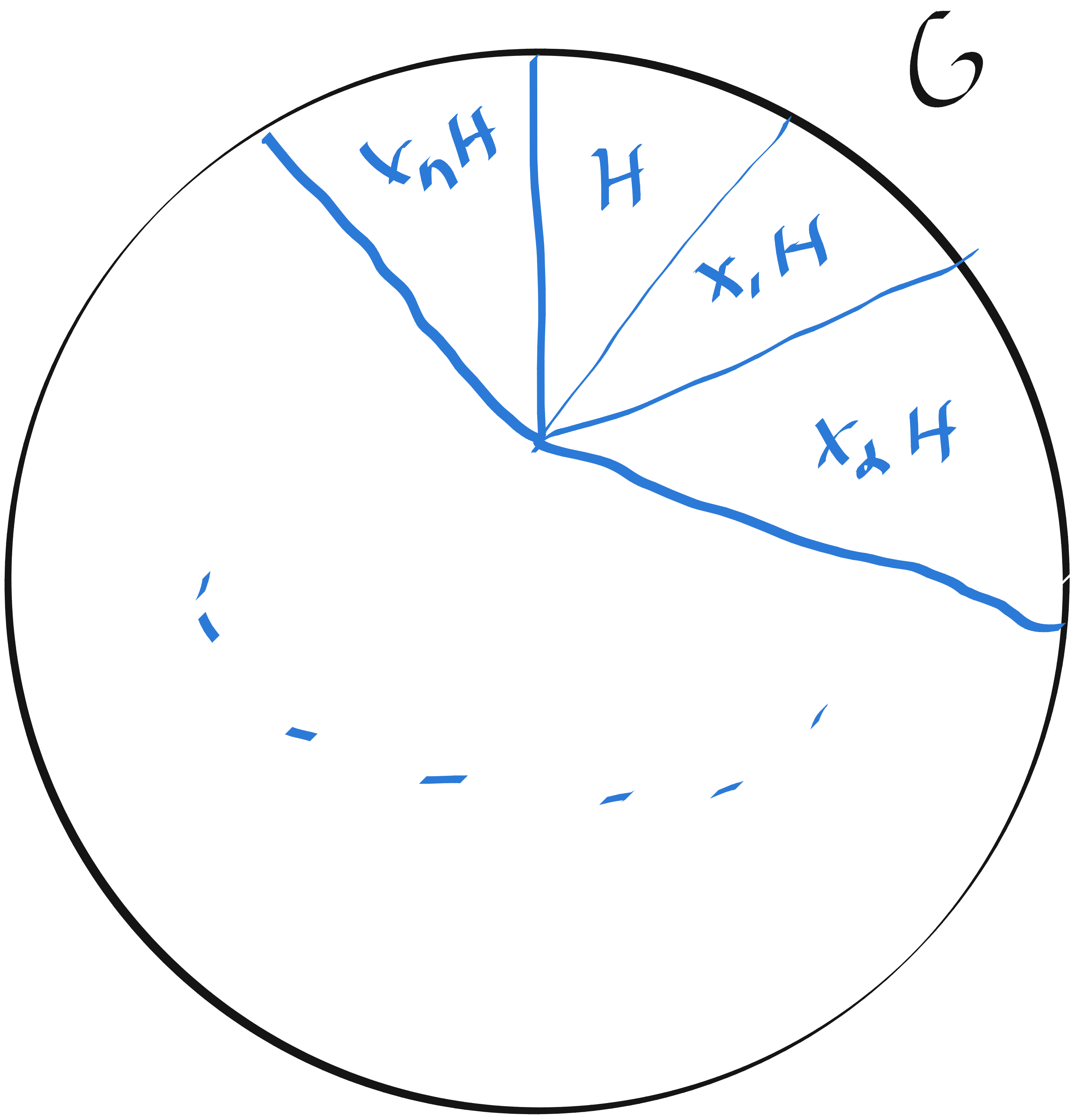
Definition: (index) Let G be a group, $H \leq G$. Then the index of H in G, denoted $[G:H]$, is the number (cardinality) of left cosets of H in G. Note by Lagrange's Theorem, if $|G| < \infty$, then

$$[G:H] = \frac{|G|}{|H|}.$$

Margret Höft Finite Index Picture

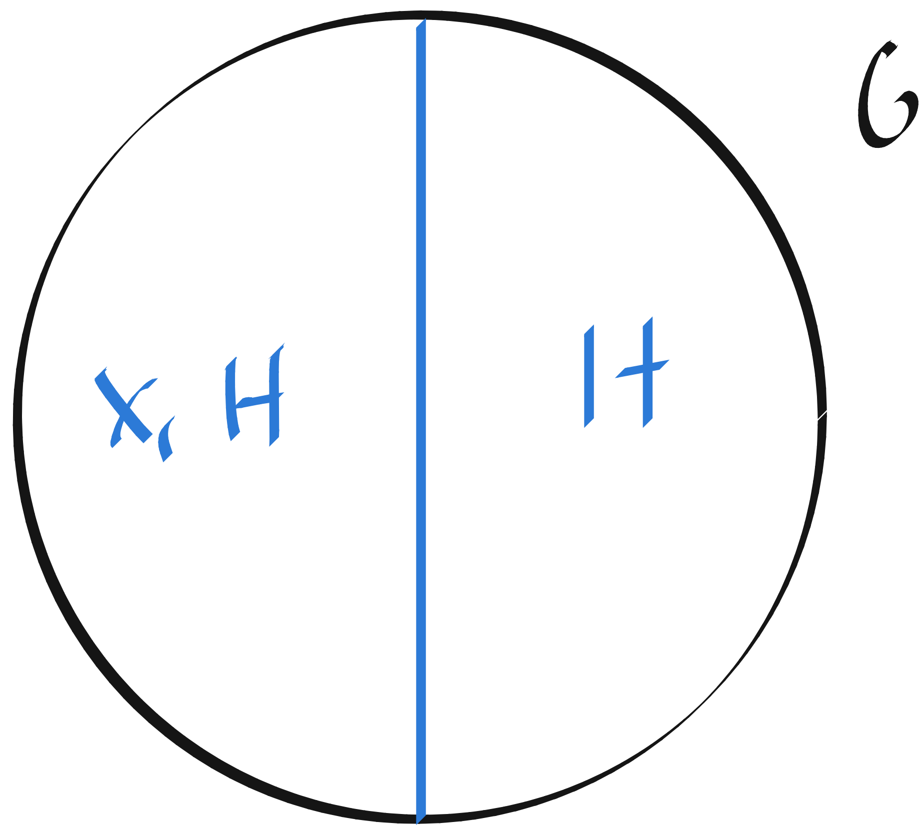
Suppose $H \leq G$ and $[G:H] < \infty$. Then we can draw:



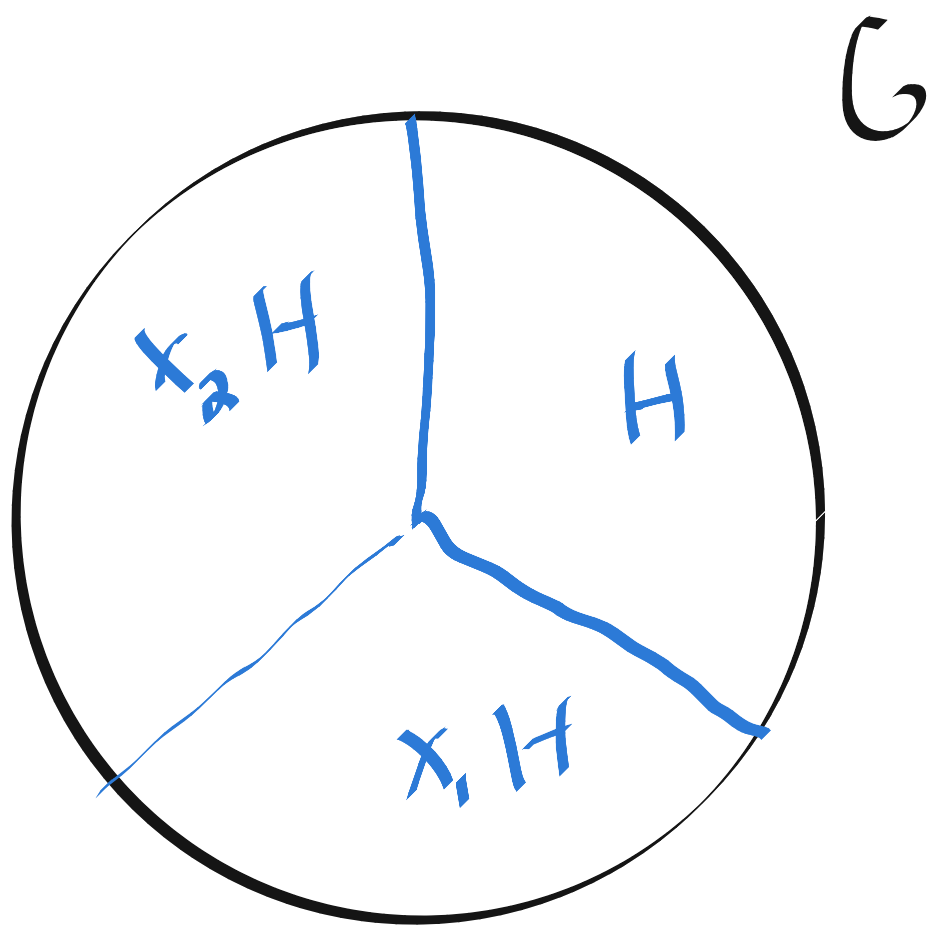
where $[G:H] = n+1$

all pieces of pie (ie. the left cosets) are of the same size and do not intersect.

$n=1$



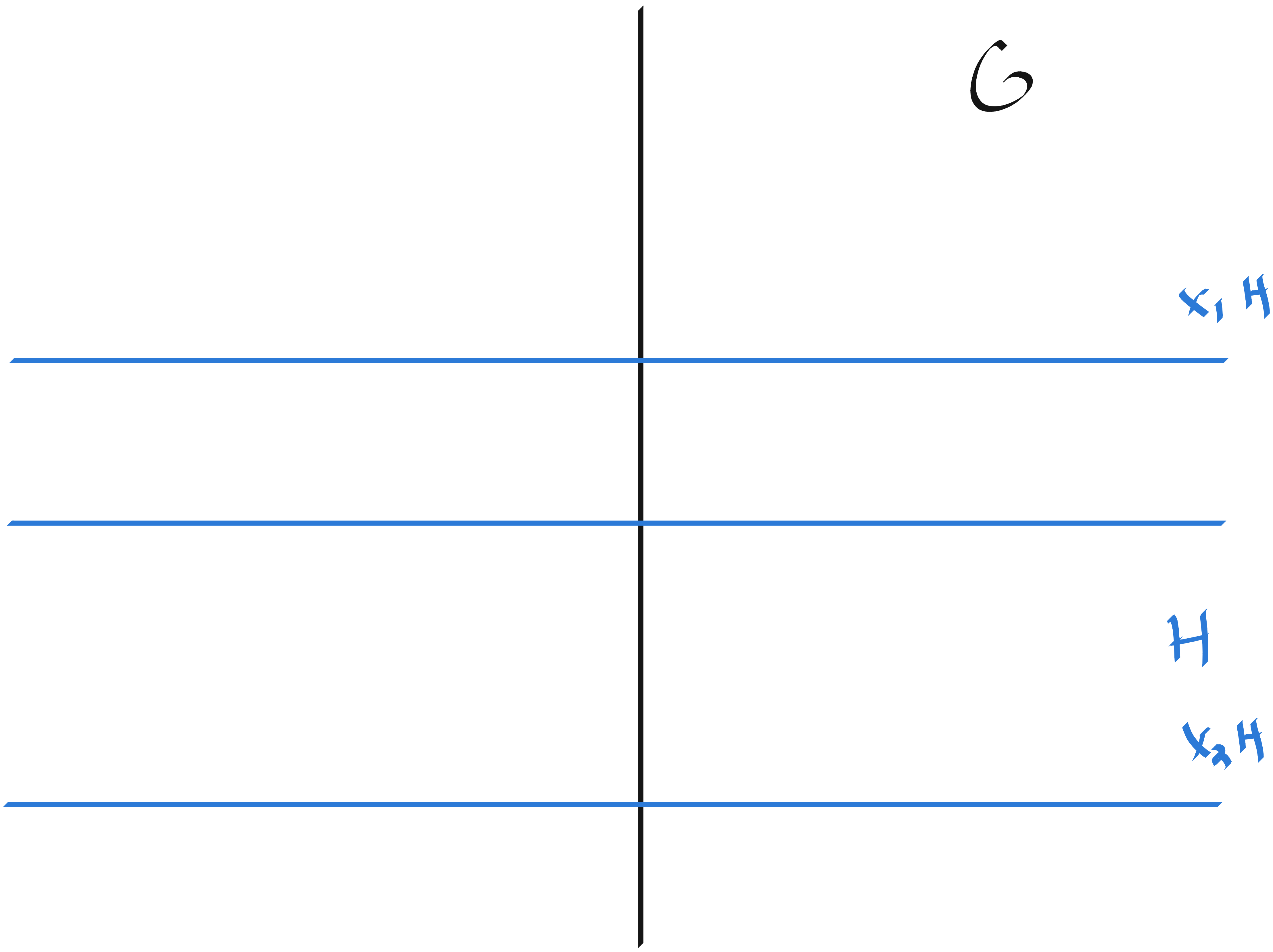
$n=2$



etc.

Frank Massey Infinite Index Picture

$$[G:H] = \infty$$



the horizontal lines are the cosets

Corollary: (groups with prime order) Suppose $|G| = p$ where p is a prime. Then G has no proper nontrivial subgroups.

proof: From Lagrange's Theorem, if $H \leq G$, $|H|$ divides $|G|$. But $|G| = p$, so the only possibilities for $|H|$ are $|H| = 1$ $(H = \{e\})$ or $|H| = p$ $(H = G)$

$\square$

Corollary: (order of an element) Let G be a group, $|G| < \infty$. Then if $x \in G$, $o(x) \mid |G|$.

proof: Recall that we defined $o(x)$ to be $|\langle x \rangle|$. By Lagrange's Theorem,

$$o(x) = |\langle x \rangle| \text{ divides } |G|.$$

$\square$

Proposition: (multiplicity of the index)

Let G be a group, $H \leq G$, $K \leq H$. Then

$$[G:K] = [G:H] \cdot [H:K]$$

proof: Count cosets, with the obvious interpretations if any of these quantities is infinite.

$\square$

Definition: (center of a group) Let G be a group. We define the center of G, denoted by $Z(G)$, to be

$$\boxed{Z(G) = \{x \in G \mid xy = yx \ \forall y \in G\}}$$

i.e. all elements of G that commute with every element of G.

Note that $Z(G)$ is always abelian and normal.

Proposition: (index-two subgroups)

Let G be a group, $H \leq G$, $[G:H]=2$.

Then $H \trianglelefteq G$.

proof: HW6!