1/24/11

Last time:

Heisenberg Group
$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$
with matrix multiplication.

This is a group.

Claim:
$$H = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$$ is a subgroup.

Check two things:
- for all $g, h \in H$, $g \cdot h \in H$
- for all $g \in H$, $g^{-1} \in H$

- let $g = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$g \cdot h = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & b+d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H$$

- If $g = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, g is invertible in the Heisenberg group.

Claim: $g^{-1} = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H$

As soon as we check this is the inverse, we've shown H is a subgroup.

Check:
$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is sufficient to show that H is a subgroup of Heisenberg group ∎

One Step Subgroup Test:

let $\emptyset \neq S \subseteq \langle G, \cdot \rangle$. then $S \leq G$ iff $\forall\, g, h \in S$, $g \cdot h^{-1} \in S$.

Pf: For you!

Most of the time you end up using the two step test...

Proposition: If $\emptyset \neq S \subseteq \langle G, \cdot \rangle$, then $\exists$ a smallest (in the sense of set inclusion) subgroup H of $\langle G, \cdot \rangle$ with $S \subseteq H$.

Proof: Let K be a subgroup of $\langle G, \cdot \rangle$ containing S. since G itself is such a subgroup, $\exists$ at least one such K. Define $H = \bigcap_{\substack{K \leq G \\ S \subseteq K}} K$.

check that H is actually a subgroup of $\langle G, \cdot \rangle$. let $g, h \in H$.

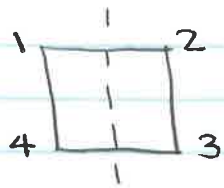
- $g \cdot h \in H$ if $g \in H$, then $g \in K$ $\forall\, K$ with $S \subseteq K$, $K \leq G$. Similarly, $h \in K$ $\forall\, K$ with $S \subseteq K$, $K \leq G$. Hence, $g \cdot h \in K$ $\forall\, K$ with $S \subseteq K$, $K \leq G$. We conclude that $g \cdot h \in \bigcap_{\substack{K \leq G \\ S \subseteq K}} K = H$

- $g^{-1} \in H$ since $g \in H$, $g \in K$ $\forall\, K$ with $S \subseteq K$, $K \leq G$. Hence, $g^{-1} \in K$ $\forall\, K$ with $S \subseteq K$, $K \leq G$. Conclude that $g^{-1} \in \bigcap_{\substack{K \leq G \\ S \subseteq K}} = H$ ■

Notation: We call the H constructed in the previous proposition the subgroup generated by S and write $H = \langle S \rangle$.

Example 1: let $\langle G, \circ \rangle$ be D_n (symmetries of a regular n-gon) let $\sigma \in D_n$ be a reflection about an axis of symmetry

eg.



reflection:

vertex 1 $\leftrightarrow$ vertex 2

vertex 4 $\leftrightarrow$ vertex 3

Applying the reflection twice gives the original picture.

Hence, if we consider $\langle \sigma \rangle \leq D_n$, then $\langle \sigma \rangle$ is isomorphic to $\mathbb{Z}_2$.

If τ is rotation by $\frac{360°}{n}$ then $\langle \tau \rangle$ is isomorphic to $\mathbb{Z}_n$. You can check that $\langle \sigma, \tau \rangle = D_n$.

Terminology (next class):

- A group $\langle G, \bullet \rangle$ is called cyclic if there exists an element $g \in G$ with $\langle g \rangle = G$.