1/28/11

Let G be a group. Then we define:

$$Z(G) = \{ g \in G \mid g \cdot x = x \cdot g \quad \forall x \in G \}$$

↑ called the center of G

Clearly $Z(G) \neq \emptyset$, $e \in Z(G)$ [$e \cdot x = x \cdot e \quad \forall x \in G$]

Remark: If G is abelian (G is commutative)
$Z(G) = G$

Example: $G = (\mathbb{R}, +)$, $Z(G) = \mathbb{R}$

$G = (GL_2(\mathbb{R}), \times)$

↑ invertible 2×2 matrix w/entries in $\mathbb{R}$.

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

$$Z(G) \supseteq \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_1 e & a_1 f \\ a_2 g & a_2 h \end{bmatrix}$$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} = \begin{bmatrix} a_1 e & a_2 f \\ a_1 g & a_2 h \end{bmatrix}$$

if $\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} \in Z(G)$ then $a_1 = a_2$

Lemma: $Z(G)$ is a subgroup of G

Pf: closure: let $x, y \in Z(G)$ WTS: $xy \in Z(G)$

$x \cdot g = g \cdot x \quad \forall g \in G$

$y \cdot g = g \cdot y \quad \forall g \in G$

WTS $(xy)g = g(xy) \quad \forall g \in G$

LHS: $(xy)g = x(yg) = x(gy)$

$= (xg)y = (gx)y = g(xy)$ ✓

Identity: $e \in Z(G)$

Inverse: $x \in Z(G)$ then $x^{-1} \in Z(G)$

$xg = gx \quad \forall g \in G$

$x^{-1}xg = x^{-1}gx$ (in G) →

$$x^{-1} \times g = x^{-1} g x$$
$$\Rightarrow e g = x^{-1} g x$$
$$\Rightarrow g = x^{-1} g x$$
$$\Rightarrow g x^{-1} = x^{-1} g x x^{-1}$$
$$\Rightarrow g x^{-1} = x^{-1} g \Rightarrow x^{-1} \in Z(G)$$

Permutations of a finite set

Let T be the set given by $\{1, 2, \dots n\}$
By a permutation of T we mean a bijection from T to T

Example: $\{1, 2, 3\}$

$1 \to 1$, $2 \to 2$, $3 \to 3$;
$1 \to 2$, $2 \to 1$, $3 \to 3$;
$1 \to 3$, $2 \to 2$, $3 \to 1$;
$1 \to 1$, $2 \to 3$, $3 \to 2$

$1 \to 2$, $2 \to 3$, $3 \to 1$;
$1 \to 3$, $2 \to 1$, $3 \to 2$

Cycle notation

Suppose under a permutation
$a_1 \to a_2$
$a_2 \to a_3$
$a_3 \to a_1$
⋮
$a_{n-1} \to a_k$
$a_k \to a_1$
then we denote it by $(a_1\ a_2\ a_3 \dots a_k)$

We say two cycles are disjoint if they do not share any entry in common.

Aside: $(1\,2) = (2\,1)$

$(1\,2\,3) \neq (1\,3\,2)$

$(3\,1\,2)$

must preserve the cycle when reordering

notation: $1 \to 1$, $2 \to 2$, $3 \to 3$ denoted $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$1 \to 2$, $2 \to 1$, $3 \to 3$ denoted $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

We denote the set of permutations of n elements by S_n

example: $S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$

lemma: S_n is a group under composition

example: $\left[\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right] (2) = (2)$

$[f \circ g](2) =$

$f(g(2)) = f(1) = 2$

Theorem: Any permutation in S_n is either the identity element or the product of disjoint cycles.

ex: $n = 7$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 6 & 1 & 3 & 4 & 7 \end{pmatrix}$

$1 \to 2 \to 5 \to 3 \to 6 \to 4 \to 1$

$(1\ 2\ 5\ 3\ 6\ 4)(7)$

1/31/11

$G = S_n$

We may write permutations using cycle notation

WTS: $\forall \sigma \in S_n$, σ may be represented as a product of disjoint cycles.

Pf: Assume S_n acts on $\{1, 2, \cdots, n\}$.

Take $\sigma \in S_n$. Let n_1 be the smallest natural number with $\sigma(n_1) \neq n_1$. (If no such n_1 exists, then σ = identity permutation) consider iterates of $\sigma^k(\sigma(n_1))$. Either these iterates contain all elements in $\{1, 2, \cdots, n\}$ that are not fixed by σ or it does not.

If iterating σ on n_1 yields all numbers not fixed by σ, then σ is a cycle and done.

If this does not occur then iterates of n_1 are some subset of the natural numbers in $\{1, \cdots, n\}$ that are not fixed by σ.

Remove the iterates of n_1 from the set $\{1, \cdots, n\}$. Choose $n_2 \in \{1, \cdots n\} \setminus \{\sigma^k(n_1)\}$ minimal such that $\sigma(n_2)$ does not fix n_2.

repeat the process until exhausted.

Ex: Choose $\sigma \in S_6$, $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 6$, $\sigma(4) = 5$, $\sigma(5) = 4$, $\sigma(6) = 1$

In disjoint cycle notation;
$\sigma = (1\ 3\ 6)(4\ 5)$ ← don't need to write (2) because it is fixed.

Observation:
If σ_1 and σ_2 are two cycles such that if $\sigma_1(k) \neq k$ then $\sigma_2(k) = k$ and $\sigma_2(k) \neq k$ implies $\sigma_1(k) = k$. Then σ_1 and σ_2 commute. We call these **disjoint** cycles.

Def: $\sigma \in S_n$ is called a **transposition** if it is a 2-cycle; i.e. there are precisely two numbers in $\{1, \dots, n\}$ that σ does not fix.

ex: $(1\ 5)$, $(2\ 7)$, $(9\ 6)$ are all 2-cycles.
But, $(1\ 2\ 3)$ is not a 2-cycle

Thm: If $\sigma \in S_n$, then σ is expressible as a product of (not necessarily disjoint) transpositions.

Before proof, **example:**

$\sigma = (1\ 5\ 7\ 9)(2\ 3\ 6) \in S_9$

$\sigma = (1\ 5)(5\ 7)(7\ 9)(2\ 3)(3\ 6)$

$= (1\ 5)(5\ 7)(2\ 3)(3\ 2)(7\ 9)(2\ 3)(3\ 6)$

$(3\ 2)(3\ 2) =$ identity

$= (1\ 5)(5\ 7)(7\ 9)(3\ 6)(6\ 2)$

Proof ↓

Pf: since all $\sigma \in S_n$ may be decomposed as products of disjoint cycles, it suffices to prove the theorem for a cycle.
If $\sigma = (n_1 n_2 \cdots n_k n_{k+1})$, $n_j \in \{1, \ldots, n\}$,
$= (n_1 n_2)(n_2 n_3) \cdots (n_{k-1} n_k)(n_k n_{k+1})$ ∎

- We have the following corollary:
If $S \subseteq S_n$ is the set of all transpositions, $\langle S \rangle = S_n$.

Def: Let $\sigma \in S_n$. We say σ is an odd permutation if σ may be expressed as the product of an odd number of transpositions.
Similarly, σ is an even permutation if it is expressed as the product of an even number of transpositions

Exs:
- $(1\ 3)$ odd
- $(1\ 2\ 3) = (1\ 2)(2\ 3)$ even
- $(1\ 7\ 3)(9\ 2\ 6)(1\ 4)$ odd
- the identity element is even
ex: $(1\ 3)(3\ 1)$

Question: Is this well-defined? (ie are there permutations which are both even and odd)
Ans: It is well-defined
there are no permutation which are both.

lemma: If $e \in S_n$ is the identity permutation then e can only be written as a product of an even number of transpositions.

Pf: It is clear that $e \neq (n_1, n_2)$ where $n_1 \neq n_2$, so you can't write e as a transposition.

Suppose $e = (n_1 n_2)(n_3 n_4) \cdots (n_k n_{k+1})$ where $(n_j n_{j+1})$ are transpositions, and $n_j \in \{1, \cdots, n\}$. Show: if you can write e as a product of m transpositions, then you can write e as the product of $(m-2)$ transpositions. (Miller, 1971)

Suppose $e = (n_1 n_2)(n_3 n_4) \cdots (n_k n_{k+1})$ and suppose $i \in \{n_1, n_2, \cdots, n_{k+1}\}$ let j denote the index of the 1st transposition, read from left to right, in which i occurs. This cannot be the final transposition, because then we wouldn't have the identity element. consider: the j^{th} transposition, then there is a $(j+1)$st transposition.

Cases:

j	$j+1$	
$(i\,m)$	$(i\,m)$	
$(i\,m)$	$(x\,y)$,	$x \neq i, m \quad y \neq i, m$
$(i\,m)$	$(i\,x)$;	$x \neq m$
$(i\,m)$	$(x\,m)$;	$x \neq i$