2/11/10

Recall: Given $H \leq G$ Define the left cosets of H in G to be sets of the form $\{gh : g \in G, h \in H\}$ for a fixed g.

Notation: gH is a left coset
$$gH = \{gh : h \in H\}$$

Ex: $G = S_3$ $\quad H = \{e, (1\,2)\}$

Choose $\sigma \in S_3$, calculate σH. If $\sigma \in H$, $\sigma H = H$.

Since $(1\,2) \cdot e = (1\,2)$ and $(1\,2)(1\,2) = e$

$\sigma \notin H$, $\sigma = (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)$

$$(1\,3)H = \{(1\,3) \cdot e, (1\,3)(1\,2)\}$$
$$= \{(1\,3), (1\,2\,3)\}$$
$$(2\,3)H = \{(2\,3) \cdot e, (2\,3)(1\,2)\}$$
$$= \{(2\,3), (1\,3\,2)\}$$
$$(1\,2\,3)H = \{(1\,2\,3) \cdot e, (1\,2\,3)(1\,2)\}$$
$$= \{(1\,2\,3), (1\,3)\}$$
$$(1\,3\,2)H = \{(1\,3\,2) \cdot e, (1\,3\,2)(1\,2)\}$$
$$= \{(1\,3\,2), (2\,3)\}$$

Observe, there are only 3 distinct left cosets. If two cosets are not equal, then they are disjoint

$$\underbrace{|S_3| = 6, \quad |H| = 2 \quad \#\text{ of left cosets} = 3}_{6 = 2 \cdot 3}$$

Similarly, if $H \leq G$ and $g \in G$, define the right coset to be $Hg = \{hg : h \in H\}$

Warning: it is not always true that $Hg = gH$

Ex: $G = S_3$; $H = \{e, (1\,2)\}$

$$(2\,3)H = \{(2\,3), (1\,3\,2)\}$$
$$H(2\,3) = \{(2\,3), (1\,2)(2\,3)\} = \{(2\,3), (1\,2\,3)\}$$

$\neq$

* ~~From now on, we only use right cosets.~~

Lemma: Let $H \leq G$ and $g, k \in G$, then either $gH \cap kH = \emptyset$ or $gH = kH$.

Pf: suppose $x \in gH \cap kH$. If this is true, then $\exists$ $h_1, h_2 \in H$ with $x = gh_1$ and $x = kh_2$.

- Show $kH \subseteq gH$:

Take $h \in H$; consider kh. WTS $\exists$ an $h' \in H$, $kh = gh'$. Since $x = kh_2$, multiplying on the right by h_2^{-1} yields $xh_2^{-1} = kh_2h_2^{-1} = k$. Substituting,
$kh = (xh_2^{-1})h = x(h_2^{-1}h)$
Since $x = gh_1$, $kh = (gh_1)(h_2^{-1}h)$
Then $kh = g(h_1h_2^{-1}h)$, set $h' = h_1h_2^{-1}h \in H$ ($h' \in H$ since $h_1, h_2, h \in H$ and H is a subgroup), then $kh = gh'$
This shows $kH \subseteq gH$. The reverse containment is proved in the same way, hence $kH = gH$ ■

Remark: $|gH| = |H|$ $\forall$ $H \leq G$, $g \in G$

Define a bijection $\varphi: H \to gH$ by $\varphi(h) = gh$

This shows $|gH| = |H|$

Prop: If $H \leq G$, then $\exists$ a subset $S \subseteq G \ni$
$G = \bigsqcup_{g \in S} gH$

Pf: Since $e_G \in H$, it is trivially true that $G = \bigsqcup_{g \in G} gH$ since $g \in gH$ $\forall g \in G$ $(g = g \cdot e_G)$
The previous thm shows only two left cosets are disjoint or equal.
Choose a representative $g \in G$ for each distinct left coset. Let
$S = \{g : g$ is a representative$\}$ Then
$G = \bigsqcup_{g \in S} gH$ ■

Thm: (Lagrange):
let $|G| < \infty$ and let $H \leq G$. Then $|H|$ divides the order of G.
Pf: Choose S a subset of G so that
$G = \bigsqcup_{g \in S} gH$ (by previous proposition)
Since $|G| < \infty$, we know S is finite.
Suppose $|S| = n$. Then
$$|G| = \left|\bigsqcup_{g \in S} gH\right| = \sum_{g \in S} |gH| = \sum_{g \in S} |H| = n|H|$$
by remark. Since $n \in \mathbb{N}$ we have $|H|$ divides $|G|$ and $\frac{|G|}{|H|} = n$ ■

Corollary: If $|G| < \infty$ and $g \in G$, then $\mathrm{ord}(g)$ divides $|G|$
Pf: $\mathrm{ord}(g) = |\langle g \rangle|$
$\langle g \rangle = H$ then $|H|$ divides $|G|$
by Lagrange's thm ■

Corollary: If $|G| = p$ where p is a prime number, then G has only $\{e_G\}$ as a proper subgroup.

Pf: Suppose $H \leq G$. Since $|H|$ divides $|G|$ by Lagrange's thm, $|H| = 1$ or p. Hence either $H = G$ or $H = \{e_G\}$ ■

Scholium: If $|G| = p$ where p is a prime, then $G \cong \mathbb{Z}_p$

* Main Thm's to know on Exam:
  - Lagrange's
  - Cauchy

Definitions & Notation:

Def: If $H \leq G$ and $g \in G$, then if $k \in gH$, we call k a representative of gH.

Notation: $[G:H]$ = the number of left cosets of H in G. We call $[G:H]$ the index of H in G. If the number of left cosets is finite just say $[G:H] = \infty$.