corrections: 2/14/11

$SL_n(\mathbb{R})$ or $SL_n(\mathbb{C})$ =
all matrices S with $\det(S) = 1$

#6 HW:

$[G \times G : H]$

$H = \{(g,g) : g \in G\}$

= "diagonal" subgroup

(1st and 2nd coordinates always equal)

Recall: For $H \leq G$ and $g \in G$ we define the left cosets of H to be $gH = \{gh : h \in H\}$ and the right cosets $Hg = \{hg : h \in H\}$

We had an example of $H \leq S_3$ where gH is not always equal to Hg.

Def: $H \leq G$ is called normal if $\forall g \in G$, $gH = Hg$.

Notation: $H \trianglelefteq G$ denotes H is normal in G

Remarks: 1) $Hg = gH$ is equivalent to $gHg^{-1} = H$ where $gHg^{-1} = \{ghg^{-1} : h \in H\}$

we'll translate between these two equivalent forms

2) Recall from HW2:

$N_G(H) = \{g \in G : gHg^{-1} = H\}$

H is normal iff $N_G(H) = G$.

3) In any group G, $H = \{e_G\}$ and $H = G$ are always normal in G. If these are the only 2 normal subgroups of G, we say G is simple.

Examples of Normal Subgroups:

1) If G is abelian, then any subgroup of G is normal: let $H \leq G$. If $g \in G$, then
$$gH = \{gh : h \in H\}$$
$$= \{hg : h \in H\} \quad \text{since } G \text{ is abelian}$$
$$= Hg$$

2) The same calculation shows that for G, $Z(G) \trianglelefteq G$.
Recall
$$Z(G) = \{h \in G : gh = hg \ \forall g \in G\}$$

3) $\underset{H}{\underset{\parallel}{SL_n(\mathbb{R})}} \leq \underset{G}{\underset{\parallel}{GL_n(\mathbb{R})}}$

$SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R})$

Pf: $SL_n(\mathbb{R}) = \{T \in GL_n(\mathbb{R}) : \det(T) = 1\}$

Take $S \in GL_n(\mathbb{R})$. WTS: $SHS^{-1} = H$

$SHS^{-1} \subseteq H$: Let $T \in SL_n(\mathbb{R})$. Then
$$\det(STS^{-1}) = \det(S)\det(T)\det(S^{-1})$$
$$= \det(S)\det(T)\frac{1}{\det(S)}$$
$$= \det(T) = 1$$
This shows $STS^{-1} \in SL_n(\mathbb{R})$ so, $SHS^{-1} \subseteq H$.

$H \subseteq SHS^{-1}$: Let $T \in SL_n(\mathbb{R})$. Then
$\det(S^{-1}TS) = 1$, so $K = S^{-1}TS \in SL_n(\mathbb{R})$
Then $T = (SS^{-1})T(SS^{-1}) = S(S^{-1}TS)S^{-1}$
$= SKS^{-1} \in SHS^{-1}$
because $K \in SL_n(\mathbb{R}) = H$. Hence $H \subseteq SHS^{-1}$.

Since $H \subseteq SHS^{-1}$ and $SHS^{-1} \subseteq H$, $H = SHS^{-1}$,
so $H \trianglelefteq G$ ∎

What are normal subgroups good for?
They are "common", for one

Def: Let G, H be groups. A map $\varphi: G \to H$ is called a Homomorphism if $\forall\, g_1, g_2 \in G$,
$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$.

Remark: φ is not necessarily bijective, so may not be an isomorphism. Examples later.

lemma: Let $\varphi: G \to H$ be a homomorphism. Then $\forall\, g \in G$,
1. $\varphi(e_G) = e_H$
2. $\varphi(g)^{-1} = \varphi(g^{-1})$

Pf: 1. Take $k \in \varphi(G)$. Then $k = \varphi(g)$ for $g \in G$. Then
$$k = \varphi(g \cdot e_G) = \varphi(g)\varphi(e_G) = k\,\varphi(e_G)$$
multiply on the left by k^{-1}:
$$k^{-1}k = k^{-1}k\,\varphi(e_G)$$
$$e_H = e_H\,\varphi(e_G) = \varphi(e_G)$$
this achieves the first statement.

2. If $g \in G$, $e_H = \varphi(e_G) = \varphi(g g^{-1}) = \varphi(g)\varphi(g^{-1})$

also, $e_H = \varphi(g^{-1}g) = \varphi(g^{-1})\varphi(g)$

So, $\varphi(g^{-1}) = \varphi(g)^{-1}$. ∎

Thm: Let $\varphi: G \to H$ be a homomorphism. Let $\ker\varphi = \{g \in G : \varphi(g) = e_H\}$. Then $\ker\varphi \trianglelefteq G$.

Pf: Step 1: Show $\ker\varphi \leq G$

1. $\ker\varphi \neq \emptyset$
2. $g, h \in \ker\varphi$, then $gh \in \ker\varphi$
3. $g \in \ker\varphi$, then $g^{-1} \in \ker\varphi$

1 $\ker\varphi \neq \emptyset$ since $\varphi(e_G) = e_H$,
so $e_G \in \ker\varphi$.

2 $g, h \in \ker\varphi$.
$\varphi(gh) = \varphi(g)\varphi(h) = e_H \cdot e_H = e_H$
so $gh \in \ker\varphi$

3 by previous lemma, if $g \in \ker\varphi$,
$e_H = \varphi(g)$, so $\varphi(g)^{-1} = e_H$, but
$\varphi(g)^{-1} = \varphi(g^{-1})$ so, $g^{-1} \in \ker\varphi$.

Next time we show that $\ker\varphi \trianglelefteq G$.