Thm: Let $\varphi: G \to H$ be a homomorphism. Let $\ker\varphi = \{g \in G : \varphi(g) = e_H\}$. Then $\ker\varphi \trianglelefteq G$.

Pf: Step 1: Show $\ker\varphi \leq G$

1. $\ker\varphi \neq \emptyset$
2. $g, h \in \ker\varphi$, then $gh \in \ker\varphi$
3. $g \in \ker\varphi$, then $g^{-1} \in \ker\varphi$

1. $\ker\varphi \neq \emptyset$ since $\varphi(e_G) = e_H$, so $e_G \in \ker\varphi$.
2. $g, h \in \ker\varphi$.
   $\varphi(gh) = \varphi(g)\varphi(h) = e_H \cdot e_H = e_H$
   so $gh \in \ker\varphi$
3. by previous lemma, if $g \in \ker\varphi$, $e_H = \varphi(g)$, so $\varphi(g)^{-1} = e_H$, but $\varphi(g)^{-1} = \varphi(g^{-1})$ so, $g^{-1} \in \ker\varphi$.

Next time we show that $\ker\varphi \trianglelefteq G$.

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Continuation of proof: prove $\ker\varphi$ is normal

Take $g \in G$, show $g(\ker\varphi)g^{-1} = \ker\varphi$.

$g(\ker\varphi)g^{-1} \subseteq \ker\varphi$

pick $h \in \ker\varphi$. Then $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1})$ (since φ is a homomorphism). So,

$$\begin{aligned}
\varphi(g)\varphi(h)\varphi(g^{-1}) &= \varphi(g)\varphi(h)\varphi(g)^{-1} \quad \text{(lemma)}\\
&= \varphi(g)\, e_H\, \varphi(g)^{-1} \quad (h \in \ker\varphi)\\
&= \varphi(g)\varphi(g)^{-1} = e_H
\end{aligned}$$

This shows $ghg^{-1} \in \ker\varphi$, hence $g(\ker\varphi)^{-1} \subseteq \ker\varphi$.

$\ker\varphi \subseteq g(\ker\varphi)g^{-1}$

observe that if $g \in G$, $\varphi(g^{-1}hg) = \varphi(g)^{-1}\varphi(h)\varphi(g)$
$= e_H \quad \forall h \in \ker\varphi$

So, $g^{-1}hg \in \ker\varphi$, set $k = g^{-1}hg$. Then

$$\begin{aligned}
h &= (gg^{-1})h(gg^{-1}) = g(g^{-1}hg)g^{-1}\\
&= gkg^{-1} \in g(\ker\varphi)g^{-1} \text{ since } k \in \ker\varphi
\end{aligned}$$

this shows $h \in g(\ker\varphi)g^{-1}$, so $\ker\varphi \subseteq g(\ker\varphi)g^{-1}$

we have $g(\ker\varphi)g^{-1} = \ker\varphi$ …

Examples:

1) $G = S_n \qquad H = \{-1, 1\} \leq \langle \mathbb{R} \setminus \{0\}, \cdot \rangle$

$\varphi : G \to H \qquad \varphi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$

Show this is a homomorphism, determine $\operatorname{Ker} \varphi$.

$\varphi(\sigma_1 \sigma_2) = \varphi(\sigma_1)\,\varphi(\sigma_2)$, A_n = all even permutations

Cases: both even, both odd, even + odd

2) $G = GL_2(\mathbb{R}) \qquad H = \langle \mathbb{R} \setminus \{0\}, \cdot \rangle$

$\varphi : G \to H \qquad \varphi(T) = \det(T)$

Show φ is a homomorphism, determine $\operatorname{Ker} \varphi$

find det

3) $G = C(\mathbb{R})$ (addition)

$= \{$continuous, real-valued functions defined on $\mathbb{R}\}$

$H = \langle \mathbb{R}, + \rangle$

for $x \in \mathbb{R}$, define $\varphi_x(f) = f(x)$.

Same question for φ_x.

4) G arbitrary, $H = \{xyx^{-1}y^{-1} : x, y \in G\}$

Show $H \trianglelefteq G$. Can you find φ with $H = \operatorname{Ker} \varphi$?

1) WTS $\Phi(\sigma_1, \sigma_2) = \Phi(\sigma_1)\,\Phi(\sigma_2)$

case 1: $\Phi(\sigma_1, \sigma_2) = \Phi(\sigma_3) = 1$ even

$\sigma_1, \sigma_2 =$ even $\quad \sigma_3 = \sigma_1 \sigma_2$

$= \Phi(\sigma_1)\,\Phi(\sigma_2)$

$= 1 \circ 1 = 1$

case 2: $\Phi(\sigma_1, \sigma_2) = \Phi(\sigma_3) = 1$

$\sigma_1, \sigma_2 =$ odd

$= \Phi(\sigma_1) \cdot \Phi(\sigma_2)$

$= -1 \circ -1 = 1$

case 3 $\Phi(\sigma_1, \sigma_2) = \Phi(\sigma_3) = -1$

$\sigma_1 =$ odd, $\sigma_2 =$ even $\quad \sigma_3 =$ odd

$\Phi(\sigma_1)\,\Phi(\sigma_2) = -1 \circ 1 = -1$

$\ker \Phi = A_n$ (all even permutations)

2) $\Phi: GL_2(\mathbb{R}) \to \langle \mathbb{R} \setminus \{0\}, \cdot \rangle \qquad \Phi(T) = \det(T)$

Φ is homomorphic

show $\forall$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$\Phi(AB) = \Phi(A)\,\Phi(B)$, $\quad \Phi(AB) = \det(AB)$

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

$\det(AB) = [(ae + fg)(cf + dh) - (af + bh)(ce + dg)]$

$= aden + bcfg - adfg - bcen$

$\det(A)\det(B) = (ad - bc)(eh - fg)$

$= aden + bcfg - adfg - bcen$

$\therefore \det(AB) = \det(A)\det(B)$

$\Phi(AB) = \Phi(A)\,\Phi(B)$

Φ is homomorphic

$\ker \Phi = \{T \in GL_2(\mathbb{R}) : \det(T) = 1\} = SL_2(\mathbb{R})$