2/4/11

Continuation of proof from last time...

Case 1:

$(m\ i)(m\ i) = e$

So we throw away the j and j+1 st 2-cycles. Reduce length of the product by 2.

Case 2:

$\overset{j}{(m\ i)}\overset{j+1}{(x\ y)} = \overset{j}{(x\ y)}\overset{j+1}{(m\ i)}$

push $(m\ i)$ one unit to the right in our product

case 3:

$\overset{j}{(m\ i)}\overset{j+1}{(m\ y)} = (y\ i\ m)$
$= (y\ i)(i\ m) = \overset{j}{(y\ i)}\overset{j+1}{(m\ i)}$

again we've pushed $(m\ i)$ one unit to the right

Case 4:

$\overset{j}{(m\ i)}\overset{j+1}{(i\ y)} = (y\ m\ i) = \overset{j}{(y\ m)}\overset{j+1}{(m\ i)}$

again we've pushed $(m i)$ one unit to the right.

In cases 2 - 4 keep pushing $(m\ i)$ one unit to the right

Eventually either have $(m\ i)(m\ i)$ [case 1] so we're done, or push $(m\ i)$ all the way to $(n_k\ n_{k+1})$ - contradiction since $m$ can't be in the last term.

Conclusion: given $e = (n_1\ n_2)(n_3\ n_4) \cdots (n_k\ n_{k+1})$ we may always reduce the numbers of 2-cycles by 2

Repeating the process we either get $e = (n_1 n_2)(n_3 n_4)$ or $e = (n_1 n_2)$. Hence, can only be writen as even perm. ■

Thm: Any permutation $\sigma \in S_n$ is either even or odd but not both.

Pf: Take $\sigma \in S_n$, suppose you can write $\sigma = (t_1\ t_2)(t_3\ t_4)\cdots(t_k\ t_{k+1})$ with the number of 2-cycles even.
By contradiction, suppose $\sigma = (s_1\ s_2)(s_3\ s_4)\cdots(s_j\ s_{j+1})$ with the number of 2-cycles odd.
Then $\sigma^{-1}$, for example, may be written as, $\sigma^{-1} = (t_k\ t_{k+1})(t_{k-1}\ t_{k-2})\cdots(t_2\ t_1)$ eve
Then $e = \sigma\sigma^{-1}$
$= (s_1 s_2)(s_3 s_4)\cdots(s_j\ s_{j+1}) \cdot (t_k\ t_{k+1})(t_{k-1-2})\cdots(t_2\ t_1)$
$=$ (even number of cycles) + (odd number of cycles)
$=$ (odd number of cycles)
Contradiction, hence $\sigma$ must be odd or even but not both. ■

Def: Let $A_n \subseteq S_n$ consist of all even permutations.

example: $S_4$

Find all even permutations.

$\{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3),$
$(1\ 2\ 3) = (1\ 2)(2\ 3), (1\ 3\ 4) = (1\ 3)(3\ 4),$
$(2\ 3\ 4) = (2\ 3)(3\ 4), (1\ 2\ 4) = (1\ 2)(2\ 4),$
$(1\ 3\ 2) = (1\ 3)(3\ 2), (1\ 4\ 3) = (1\ 4)(4\ 3)$
$(2\ 4\ 3) = (2\ 4)(4\ 3), (1\ 4\ 2) = (1\ 4)(4\ 2)\} = A_4$

This is the whole list because in $S_4$, an even permutation can either be the identity or reduce to a product of two transpositions.

Thm: $A_n \leq S_n$ ( $A_n$ = "alternating" group of permutations)

Pf: let $\sigma, \tau \in A_n$

Show 1) $\sigma^{-1} \in A_n$

2) $\sigma\tau \in A_n$

1) We showed in the proof that even/odd notation is well-defined that if

$\sigma = (t_1\ t_2)(t_3\ t_4) \cdots (t_k\ t_{k+1})$ then

$\sigma^{-1} = (t_k\ t_{k+1})(t_{k-1}\ t_{k-2}) \cdots (t_2\ t_1)$

So this implies that $\sigma$ is even iff $\sigma^{-1}$ is even.

2) $\sigma = (t_1\ t_2)(t_3\ t_4) \cdots (t_k\ t_{k+1})$

$\tau = (s_1\ s_2)(s_3\ s_4) \cdots (s_j\ s_{j+1})$

$\sigma\tau = (t_1 t_2)(t_3\ t_4) \cdots (t_k\ t_{k+1}) \cdot (s_1 s_2)(s_3 s_4) \cdots (s_j s_{j+1})$

= (even number)+(even number)

=(even number of transpositions)

So $\sigma\tau \in A_n$ ∎

There are precisely $\frac{n!}{2}$ elements in $A_n$.

Thm: (Cayley) Any group is isomorphic to a subgroup of some symmetric group. (the group of all bijections on a set w/ function composition)

Remarks:

1) G any group, G may not be isomorphic to the full symmetric group

2) Thm is really only applied when G is finite

proof →

Pf: Assume $G$ is finite. Then $G$ has $n$ elements $\{g_1, g_2, \dots, g_n\}$. If $g \in G$, then
$g \cdot g_k = g_j$ $\forall k$, $1 \le k \le n$ and some $j$, $1 \le j \le n$
If $k_1 \ne k_2$ then if $g \cdot g_{k_1} = g \cdot g_{k_2}$,
multiply on the left by $g^{-1}$ to get
$g_{k_1} = g_{k_2}$, contradiction.
Show: If $g g_{k_1} = g_{j_1}$ and $g g_{k_2} = g_{j_2}$
then $j_1 = j_2$ iff $k_1 = k_2$.
permutation $j \mapsto k$.