2/9/11

Recall: $g \in G$ we define the order of g to be $|\langle g \rangle| := \text{ord}(g)$.
$\text{ord}(g)$ = the minimal $n \in \mathbb{N}$ with $g^n = e_G$

Thm: let $\varphi \in G \to H$ be an isomorphism. Then for all $g \in G$, $\text{ord}(g) = \text{ord}(\varphi(g))$.

Pf: Since φ is an isomorphism,

$$\varphi(g^K) = \varphi(\underbrace{g \cdot \cdots \cdot g}_{K})$$
$$= \overbrace{\varphi(g) \cdots \varphi(g)} = (\varphi(g))^K \quad \forall K \in \mathbb{N}$$

Since φ is injective,

$$e_H = (\varphi(g))^K = \varphi(g^K)$$
$$\Rightarrow g^K = e_G$$

This implies $\text{ord}(\varphi(g)) \geq \text{ord}(g)$

But $\underset{\varphi(e_G) = e_H}{\underset{\shortparallel}{\varphi(g^n)}} = (\varphi(g))^n \quad (n = \text{ord}(g))$

This implies $\text{ord}(\varphi(g)) \leq \text{ord}(g)$

so, $\text{ord}(\varphi(g)) = \text{ord}(g)$ ■

Application: To show 2 groups are not isomorphic show that one has an element with order that does not occur in the others.

Notation: G isomorphic to H written as $G \cong H$
Not isomorphic : $G \ncong H$

Examples: 1) $G = \langle \mathbb{R}, + \rangle \qquad H = \langle \mathbb{R} \setminus \{0\}, \cdot \rangle$
H has an element of order 2 $(g = -1)$
but G has no such element.
$H \ncong G$

2) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad H = \mathbb{Z}_4$

If $g \in G$, $g \neq (0,0)$

$\text{ord}(g) = 2$

$(g = (1,1), (1,0), (0,1))$

But $1 \in H$, $\text{ord}(1) = 4$ ~~So,~~ $H \not\cong G$

3) $G = D_6$ (symmetries of a regular hexagon)

$H = A_4$ ($A_4 \leq S_4$ all even permutations)

$|H| = \frac{4!}{2} = 12$

$D_6 =$ 6 rotations, 6 reflections

$|D_6| = 12$

$$\left.\begin{matrix} (1\ 2)(3\ 4) \\ (1\ 3)(2\ 4) \\ (1\ 4)(2\ 3) \end{matrix}\right\} \text{All products of disjoint 2-cycles in } A_4$$

$\text{ord}(\sigma_1) = \text{ord}(\sigma_2) = \text{ord}(\sigma_3)$

$\sigma_1 \sigma_1 = ((1\,2)(3\,4))((1\,2)(3\,4))$

$= (1\,2)\underbrace{(3\,4)(3\,4)}_{e}(1\,2)$

$= (1\,2)(1\,2) = e$

$\therefore$ There are 3 elements of order 2 in H

There are 6 elements of order 2 in G

$\therefore H \not\cong G$

MORE EXAMPLES INVOLVING ORDER

4) $G = \mathbb{Z}_5$

$g \in G$, $g \neq 0$

$\text{ord}(g) = 5$

check: $g = 1, 2, 3, 4$

and add until you get to zero

Also: $G = \mathbb{Z}_p$, p is prime

$g \in G$, $g \neq 0$

$\text{ord}(g) = p$

5) $G = \mathbb{Z}_6$

$g \in G, \quad g \neq 0$

$g = 1, 2, 3, 4, 5$

$\text{ord}(1) = 6$

$\text{ord}(2) = 3$

$\text{ord}(3) = 2$

$\text{ord}(4) = 3$

$\text{ord}(5) = 6$

Conjecture: If $|G| < \infty$, then if $g \in G$, $\text{ord}(g)$ divides $|G|$

6) $G = \mathbb{Z}$

If $g \neq 0$, $\text{ord}(g) = \infty$

7) Notation: general linear group $\left\{ \begin{array}{l} \bullet\ GL_n(\mathbb{C}) \quad (\text{or } GL_n(\mathbb{R})) \\ \text{denotes all invertible } n \times n \text{ matrices} \\ \text{with entries in } \mathbb{C} \ (\text{or } \mathbb{R}) \end{array} \right.$

special linear group $\left\{ \begin{array}{l} \bullet\ SL_n(\mathbb{C}) \leq GL_n(\mathbb{C}) \\ SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : |\det(A)| = 1\} \end{array} \right.$

Show: $GL_n(\mathbb{C})$ contains elements of any order.

Set $n = 2$.

$$A = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & 1 \end{pmatrix} \qquad A^k = \begin{pmatrix} e^{\frac{2\pi i k}{n}} & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{ord}(A) = n$$

$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ has $\text{ord}(A) = \infty$

So $\exists$ elements in $GL_2(\mathbb{C})$ of any order.

By including $GL_n(\mathbb{C})$ into $GL_{n+1}(\mathbb{C})$ by $A = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, we get the result for $GL_n(\mathbb{C})$.

Goal: Prove the conjecture that if $|G| < \infty$ then $\text{ord}(g)$ divides $|G|$ $\forall g \in G$

We'll prove:

Lagrange's Thm: If $|G| < \infty$ and $H \leq G$ then $|H|$ divides $|G|$.

Corollary: If $g \in G$, $\text{ord}(g)$ divides $|G|$

Pf: $\text{ord}(g) = |\langle g \rangle|$. If $H = \langle g \rangle$, $|H|$ divides $|G|$, but $|H| = \text{ord}(g)$ ∎

We'll prove Lagrange's Thm by using the concepts of a coset.

Def: If $H \leq G$, define the left cosets of H in G to be sets of the form $\{gh : h \in H, g \in G\}$ for a fixed $g \in G$

Notation: $gH = \{gh : h \in H\}$

gH is a left coset

Idea: 1) $|gH| = |H|$

2) If $g, k \in G$, either $gH = kH$ or $(gH) \cap (kH) = \emptyset$

$G = S_3$

$H = \{e, (1\,2)\}$

determine all left cosets of H!