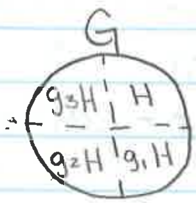
3) $G = SL_n(\mathbb{Z})$ for n even
$H = \{I_n, -I_n\}$
$H \trianglelefteq G$ since $H = Z(G)$
G/H is called $PSL_n(\mathbb{Z})$, the projected special linear group.

3/11/11

PICTURES for COSETS

Margret Hoft:

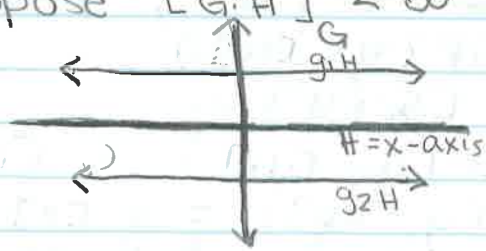
Suppose $[G:H] < \infty$, for example $[G:H] = 4$



$[G:H] = 6$

Frank Massey:

Suppose $[G:H] < \infty$



cosets are horizontal lines.

More Examples:

1) $G = \mathbb{R}$, $H = \mathbb{Z}$. Find the isomorphism class of G/H.

Recall: $h = ([0,1), \dot{+})$ $\quad x, y \in [0,1)$

$$x \dot{+} y = \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

→

cont 1) Define $\varphi : G \to K$ by $\varphi(x) = x - [x]$ if $x \geq 0$

$[x]$ = the largest integer smaller than x

ex: $[\pi] = 3$

$\pi - [\pi] = .14159\ldots$

we need φ to be a homomorphism.

In particular, $\varphi(0) = 0$

$\varphi(k-k) = 0$

$\varphi(k) - \varphi(k) = 0$ (inverses go to inverses

$x = -.15 \Rightarrow [x] = -1$

$x - [x] = -.15 - (-1) = .81$

same definition works for negative numbers

Define φ in general by $\varphi(x) = x - [x]$

Show that φ is a homomorphism:

let $x, y \in \mathbb{R}$, write $x - [x]$, $y - [y]$

consider $(x - [x]) \dot{+} (y - [y])$

$\stackrel{?}{=} (x+y) - [x+y]$

case 1) suppose $(x - [x]) + (y - [y]) \in [0,1)$

Then $(x - [x]) \dot{+} (y - [y]) = x - [x] + y - [y]$

$\stackrel{?}{=} (x+y) - [x+y]$

$x = [x] + (x - [x])$, $y = [y] + (y - [y])$

$x + y = [x] + [y] + (x - [x] + y - [y])$

$[x+y] = [x] + [y]$. Since,

$0 \leq (x - [x] + y - [y]) < 1$

Hence $(x - [x]) \dot{+} (y - [y])$

$= x + y - [x] - [y] = x + y - [x+y]$

so $\varphi(x) \dot{+} \varphi(y) = \varphi(x+y)$ if

$x - [x] + y - [y] \in [0,1)$.

$\to$

case 2) Suppose $(x-[x])+(y-[y]) \in [1,2)$
Then, $(x-[x]) \dotplus (y-[y]) = (x-[x])+(y-[y])$

$$x+y = [x]+[y]+(x-[x]+y-[y])$$
$$= [x]+[y]+1\,(x-[x]+y-[y]-1)$$
$$= [x]+[y]+1+(x-[x] \dotplus y-[y])$$

$\varphi(x+y) = x+y-[x+y] = x-[x] \dotplus y-[y]$
$\varphi(x) \dotplus \varphi(y) = (x-[x]) \dotplus (y-[y])$
Then $\varphi(x) \dotplus \varphi(y) = \varphi(x+y)$
So φ is a homomorphism

By First Isomorphism Thm, $\mathbb{R}/\ker\varphi \cong ([0,1), \dotplus)$
since φ is surjective ($\varphi([0,1)) = [0,1)$)

$$\ker\varphi = \{x \in \mathbb{R} : \varphi(x) = 0\}$$
$$= \{x \in \mathbb{R} : x-[x] = 0\}$$
$$= \{x \in \mathbb{R} : x = [x]\} = \mathbb{Z}$$

So, $\mathbb{R}/\mathbb{Z} \cong ([0,1), \dotplus)$.
Remark: $\mathbb{R}/\mathbb{Z}$ is also isomorphic to $\{z \in \mathbb{C} : |z| = 1\}$ with multiplication as group operation (circle or torus group).

FACTS (may prove)

Recall: If $\varphi : G \to K$ is a homomorphism, we show $\ker\varphi \trianglelefteq G$

Converse: If $H \trianglelefteq G$, then $\exists$ K and a homomorphism $\varphi : G \to K$ with $\ker\varphi = H$.

Sketch of Pf: $K = G/H$
$$\varphi(g) = gH$$

Goal: $H \trianglelefteq G$, want to show (maybe) $G \cong H \times G/H$
This in unfortunately false

CE $\rightarrow$

counterexample:

$G = S_3$

$H = A_3 = \langle (1\,2\,3) \rangle$

$A_3 \cong \mathbb{Z}_3$

$|G/H| = \frac{|G|}{|H|} = \frac{6}{3} = 2$

$G/H \cong \mathbb{Z}_2$

$H \times G/H \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ which is abelian and so cannot be isomorphic to S_3!

However, the thm works for abelian groups! i.e. if G is abelian, $H \leq G$, then

$$G \cong H \times G/H.$$

Fundamental Thm of Finite Abelian Groups.

Suppose G is abelian. Then $\exists$ primes, $p_1, \dots, p_n$ and numbers

$$\begin{matrix} \alpha_{1,1}, & \cdots, & \alpha_{1,n} \\ \alpha_{2,1}, & \cdots, & \alpha_{2,n} \\ \vdots & & \vdots \\ \alpha_{n,1}, & \cdots, & \alpha_{n,n} \end{matrix}$$

with
$$G \cong \mathbb{Z}_{p_1^{\alpha_{1,1}}} \times \mathbb{Z}_{p_1^{\alpha_{1,2}}} \times \cdots \times \mathbb{Z}_{p_1^{\alpha_{1,k_1}}}$$
$$\times \mathbb{Z}_{p_2^{\alpha_{2,1}}} \times \cdots \times \mathbb{Z}_{p_2^{\alpha_{2,k_2}}} \times$$
$$\times \cdots \times \mathbb{Z}_{p_n^{\alpha_{n,1}}} \times \cdots \times \mathbb{Z}_{p_n^{\alpha_{n,k_n}}}$$