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Subring Test:

$\emptyset \neq S \subseteq R$, R is a ring. S is a subring of R iff $\forall\ x, y \in S$, $x - y \in S$ and $x \cdot y \in S$ ($-y$ = additive inverse of y)

Pf: exercise!

Example from last time:

$R = M_2(\mathbb{R})$ $\quad S = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}$

show S is a subring of R.

use subring test:

let $A = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$ be elements of S

Is $A - B$, $A \cdot B \in S$? YES.....

$$A - B = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x-y & 0 \\ 0 & 0 \end{pmatrix} \in S$$

$$A \cdot B = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & 0 \end{pmatrix} \in S$$

Since $S \neq \emptyset$, S is a subring of $M_2(\mathbb{R})$ by the subring test.

Examples of Subrings:

1) $R = \mathbb{Z}$, $S = n\mathbb{Z} = \{ n \cdot k \mid k \in \mathbb{Z} \}$
for n a fixed natural number (or $n=0$)

Check that S is a subring:

$S \neq \emptyset$. Let $m, l \in S$ then $m = nk_1$ and $l = nk_2$ for some k_1, k_2 integers.

$m - l = nk_1 - nk_2 = n(k_1 - k_2) \in S$ since $k_1 - k_2 \in \mathbb{Z}$

$m \cdot l = (nk_1)(nk_2) = n(k_1 \cdot n \cdot k_2) \in S$ since $k_1 \cdot n \cdot k_2 \in \mathbb{Z}$

S is a subring of $\mathbb{Z}$ by the subring test.

Remark: since the only subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for some $n \in \mathbb{N} \cup \{0\}$, the only subrings of $\mathbb{Z}$ are also of this form.

2) $R = \mathbb{Z}[x] = \{$polynomials in x with integer coefficients$\}$

$S = \{p \in \mathbb{Z}[x] : p$ has no constant term$\}$

e.g. $p(x) = x^3 + x^2 \in S$ but

$q(x) = x^3 + x^2 + 1 \notin S$ [since 1 is constant]

check that S is a subring:

let $p(x) = \sum_{n=1}^{k} \alpha_n x^n \in S$, $q(x) = \sum_{m=1}^{l} \beta_m x^m \in S$

α_n's and β_m's $\in \mathbb{Z}$. suppose $k \geq l$

- $p(x) - q(x) = \sum_{m=1}^{l} (\alpha_m + \beta_m) x^m + \sum_{m=l+1}^{k} \alpha_m x^m$

Since always $m > 0$, $p(x) - q(x)$ has no constant term, so, $p(x) - q(x) \in S$

- $p(x) \cdot q(x) = \sum_{n=1}^{k} \sum_{m=1}^{l} \alpha_n \beta_m x^{n+m}$

now $n + m \geq 2$, so $p(x) q(x)$ has no constant term, so $p(x) q(x) \in S$

Then S is a subring of $\mathbb{Z}[x]$.

Remark: $S = \{p \in \mathbb{Z}[x] : p(0) = 0\}$

3) $R = M_n(\mathbb{R})$, $S = \{$upper triangular matrices$\}$

S is a subring of $M_3(\mathbb{R})$.

Homework!

4) $R = C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \lim_{|x| \to \infty} f(x) = 0\}$

$S = C_{00}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \exists\, y \geq 0$ with $f(x) = 0 \;\forall x, |x| \geq y\}$

Note: $f(x) = e^{-x^2}$, $f \in C_0(\mathbb{R})$ [b/c $\lim = 0$]

But $f(x) \neq 0$ for all $x \in \mathbb{R}$

$$f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$f \in C_{00}(\mathbb{R})$ since $f \equiv 0$ when $|x| > 1$.
Suppose $f, g \in C_{00}(\mathbb{R})$. Then $\exists\, y \ge 0$ and $z \ge 0$ with $f(x) = 0 \ \forall x, |x| \ge y$ and $g(x) = 0 \ \forall x, |x| \ge z$
observe, $-g(x)$ satisfies $-g(x) = 0 \ \forall x, |x| \ge z$
consider $(f-g)(x) = 0$ for $w = \max\{y, z\}$
Since then $(f-g)(x) = f(x) - g(x) = 0 - 0 = 0$
Since $|x| \ge y$ and $|x| \ge z$.
So $\forall x, |x| \ge w$, $(f-g)(x) = 0$, so $(f-g) \in C_{00}(\mathbb{R})$
for $(f \cdot g)(x)$, if $|x| \ge \min\{y, z\}$ then $(f \cdot g)(x) = f(x) \cdot g(x) = 0$ since one of $f(x)$ or $g(x)$ is equal to zero.
with $w = \min\{y, z\}$, $(f \cdot g)(x) = 0 \ \forall x$ with $|x| \ge w$, so $f \cdot g \in C_{00}(\mathbb{R})$, so $C_{00}(\mathbb{R})$ is a subring of $C_0(\mathbb{R})$.

5) $R = \mathbb{Z}_n$ Then $\mathbb{Z}_n$ may have no nontrivial subrings!
Always has $S = \mathbb{Z}_n$ and $S = \{0\}$ as subrings (trivial)
If n is prime, then $\mathbb{Z}_n$ has no nontrivial subrings (by Lagranges Thm - any subgroup must have order dividing n)
But for example, since $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ (as groups) we can take $\mathbb{Z}_2 \times \{0\} = S$ as a subgroup of $\mathbb{Z}_6$

<u>Def:</u> If R_1, R_2 are rings, we can define the direct product $R = R_1 \times R_2$ as the ring with operations on ordered pairs (x, y) $x \in R_1$, $y \in R_2$ by $(x, z \in R_1,\ y, w \in R_2)$

$$(x, y) + (z, w) = (x+z, y+w)$$
$$(x, y) \cdot (z, w) = (x \cdot z, y \cdot w)$$