Last time: 3/21/11

$R, S$ are rings, we define the direct product $R \times S$ to be the ring with operations ($x_1, x_2 \in R$, $y_1, y_2 \in S$)

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2)$

$R \times \{0\}$ should be "the same" ring as $R$, and $\{0_R\} \times S$ should be "the same" ring as $S$. What does same mean?

Def: Let $R, S$ be rings. $\varphi : R \to S$ is a ring homomorphism if $\forall x, y \in R$, $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$

Note: $\varphi(0_R) = 0_S$ but not necessarily the case that if $R$ is unital, $\varphi(1_R)$ need be a unit for $S$

Def: $\varphi$ is an isomorphism if $\varphi$ is a bijective ring homomorphism

Define for $R$ and $S$ rings,
$\varphi : R \to R \times S$ by $\varphi(x) = (x, 0_S)$ $\forall x \in R$

Pf: let $x, y \in R$. check $\varphi$ is a homomorphism:
$\varphi(xy) = (xy, 0_S) = (x, 0_S)(y, 0_S) = \varphi(x)\varphi(y)$
$\varphi(x+y) = (x+y, 0_S) = (x, 0_S) + (y, 0_S) = \varphi(x) + \varphi(y)$
$\varphi$ is a homomorphism, surjective onto its image
$\varphi(R) = R \times \{0_S\}$
Is $\varphi$ injective? Suppose $\varphi(x) = \varphi(y)$ then
$(x, 0_S) = (y, 0_S)$ so, $(x-y, 0_S) = (0_S, 0_R)$
$\Rightarrow x - y = 0_R$ so, $x = y$
$\varphi$ is an isomorphism from $R$ onto $R \times \{0_S\}$ Hence $R$ is isomorphic to a subring of $R \times S$.

Similarly, $S$ is isomorphic to a subring of $R \times S$ (namely, $\{0_R\} \times S$) by defining
$\psi: S \to R \times S$, $\psi(x) = (0_R, x)$ $\forall$ $x \in S$

However, $R \times \{0_S\}$ and $\{0_R\} \times S$ have special properties, not common to ordinary subrings.

Take $x \in R$, $y \in S$. Suppose $z \in R$.
$(x, y) \cdot (z, 0_S) = (xz, y0_S)$
$= (xz, 0_S)$ if $y \cdot 0_S = 0_S$ $\forall$ $y \in S$
$\in R \times \{0_S\}$

Similarly, if $w \in S$,
$(x, y) \cdot (0_R, w) = (0_R, y \cdot w) \in \{0_R\} \times S$
again providing $x \cdot 0_R = 0_R$

Prop: Let $R$ be a ring, $x, y \in R$. Then
i) $x \cdot (-y) = -xy$
ii) $x \cdot 0_R = 0_R = 0_R \cdot x$

Pf:
ii) $x \cdot x + x \cdot 0_R = x \cdot (x + 0_R)$ (by distributivity)
$= x \cdot x = x \cdot x + 0_R$
subtracting $x \cdot x$ from both sides we have $x \cdot 0_R = 0_R$
Similarly, $x \cdot x + 0_R \cdot x = (x + 0_R) \cdot x = x \cdot x = x \cdot x + 0_R$
So, $0_R \cdot x = 0_R$

i) Recall by unique element in $R$ with $xy + (-xy) = 0_R$
If we show $xy + (x \cdot (-y)) = 0_R$ then by uniqueness, $-xy = x \cdot (-y)$,
$xy + (x \cdot (-y)) = x \cdot (y + -y)$ (distributive)
$= x \cdot (0_R) = 0_R$ by ii)
Hence, $xy + (x \cdot (-y)) = 0_R$
So, $x \cdot (-y) = -xy$ ■

The proposition shows that $\forall x \in R$, $y \in S$, $z \in R$, $w \in S$, $(x,y)\cdot(0_R, w) \in \{0_R\} \times S$ and $(x,y)(z, 0_S) \in R \times \{0_S\}$.

Similarly, $(0_R, w)\cdot(x,y) \in \{0_R\} \times S$ and $(z, 0_S)(x,y) \in R \times \{0_S\}$

EX: $R = M_2(\mathbb{R})$, $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$

S is a subring of R, but that does not have the property that $\forall x \in S$, $T \in R$, $x \cdot T \in S$.

$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$x \cdot T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin S$

Similarly, not true that $\forall w \in R$, $y \in S$, $w \cdot y \in S$

$y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $w = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$w \cdot y = w \notin S$

Def. R is a ring, $S \subseteq R$ a subring. S is called an <u>ideal</u> of R if $\forall x \in R$, $y \in S$, $x \cdot y \in S$ and $y \cdot x \in S$.

Note: If only $x \cdot y \in S$ for $x \in R$, $y \in S$ we say S is a <u>left</u> ideal.
If only $y \cdot x \in S$ for $x \in R$, $y \in S$ we say S is a <u>right</u> ideal.

If the multiplication on R is commutative then:
left ideals = right ideals = ideals

<u>Examples</u>

1) R and $\{0_R\}$ are always ideals of R.
These could be the only ideals of R (see HW on $M_n(\mathbb{R})$)

→

Examples cont

2) $R \times \{0_S\}$ and $\{0_R\} \times S$ are ideals of $R \times S$.

3) $R = C_0(\mathbb{R})$, $S = C_{00}(\mathbb{R})$
let $f \in C_0(\mathbb{R})$, $g \in C_{00}(\mathbb{R})$ then $\exists$
$y \geq 0$ with $g(x) = 0$ $\forall$ $x$ with $|x| > y$
For all such $x$, $(f \cdot g)(x) = f(x)g(x) = 0$
So, $(f \cdot g)(x) = 0$ $\forall |x| > y$, hence
$f \cdot g \in C_{00}(\mathbb{R})$ since $g \cdot f = f \cdot g$, we
have that $C_{00}(\mathbb{R})$ is an ideal in $C_0(\mathbb{R})$.