Examples cont

2) $R \times \{0_S\}$ and $\{0_R\} \times S$ are ideals of $R \times S$.

3) $R = C_0(\mathbb{R})$, $S = C_{00}(\mathbb{R})$
let $f \in C_0(\mathbb{R})$, $g \in C_{00}(\mathbb{R})$ then $\exists$ $y \geq 0$ with $g(x) = 0$ $\forall$ x with $|x| > y$
For all such x, $(f \cdot g)(x) = f(x) g(x) = 0$
So, $(f \cdot g)(x) = 0$ $\forall |x| > y$, hence $f \cdot g \in C_{00}(\mathbb{R})$ since $g \cdot f = f \cdot g$, we have that $C_{00}(\mathbb{R})$ is an ideal in $C_0(\mathbb{R})$.

* Homework #6 due wednesday 3/23/11

more...
Examples

1) $R = \mathbb{Q}$. This is a ring with usual multiplication and addition.
multiplicative unit: 1
additive identity: 0
commutative ring

Let $I \subseteq \mathbb{Q}$ be an ideal in $\mathbb{Q}$. Then either $I = \mathbb{Q}$ or $I = \{0\}$. Suppose $x \in I$, $x \neq 0$
Then $\frac{1}{x} \in \mathbb{Q}$. Since I is an ideal, $\frac{1}{x} \cdot x = 1 \in I$
By homework, $1 \in I \Rightarrow I = \mathbb{Q}$

2) $R = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \mathbb{Z} \right\}$

A noncommutative ring with the operations of matrix multiplication and addition.

unit: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

identity: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$I = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ; a_{ij} \text{ is even} \right\}$

let $T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in I$, $S = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in R$

$$TS = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$a_{ij} \cdot b_{\ell m}$ is even for all i, j, ℓ, m and the sum of every number is even so $TS \in I$.
Similarly $ST \in I$, so I is an ideal (we already showed it is a subring) in R.

<u>Def</u>: Let R be a ring and $I \subsetneq R$ be an ideal. I is called a <u>maximal ideal</u> if for all ideals J with $I \subseteq J \subseteq R$ either $J = I$ or $J = R$.

<u>Examples</u>

1) $R = \mathbb{Z}$. All ideals are of the form $n\mathbb{Z}$ for some $n \in \mathbb{N} \cup \{0\}$.

$2\mathbb{Z}$ is a maximal ideal

If $J \supseteq 2\mathbb{Z}$, then $J \neq 2\mathbb{Z} \Rightarrow J$ contains an odd number k. Then $k+1$ is even, and since J is a subring of $\mathbb{Z}$, $k+1-k = 1 \in J$
This implies $J = \mathbb{Z}$.

In general, if p is prime then $p\mathbb{Z}$ is a maximal ideal. Suppose $J \supsetneq p\mathbb{Z}$. Then $\exists$ an element $k \in J \ni p$ doesn't divide k

p prime $\Rightarrow$ $\gcd(p, k) = 1$
By the Euclidean algorithm, $\exists$ $n, m \in \mathbb{Z}$
with $np + mk = 1$
J an ideal and $p, k \in J$ $\Rightarrow$ np and mk
are elements of J. Hence $np + mp = 1 \in J \Rightarrow J = \mathbb{Z}$

2) Let $R = C_0(\mathbb{R})$, $I = C_{00}(\mathbb{R})$
Is I maximal in R? NO!

<u>Remark</u>: Any proper ideal $I \subsetneq R$ is contained in a maximal ideal.
Proof is a consiquence of Zorn's Lemma
(Aiom of Choice, Hausdorff
Maximality Principle)

<u>Connection btw Ideals & Ring Homomorphism</u>

Let R be a ring, $I \subseteq R$ an ideal. I is a subring of R. $(R, +)$ is an abelian group, so $I \trianglelefteq (R, +)$. we can then talk about the quotient group R/I

$$R/I = \left\{ \begin{array}{l} \text{left cosets of } I \text{ with operation } (x, y \in R) \\ (x + I) + (y + I) = (x + y) + I \end{array} \right\}$$

we can put a ring structure on R/I, which we already know is an abelian group, by for $x, y \in R$, $(x + I)\cdot(y + I) = (x \cdot y) + I$.
let $s, t \in I$. $(x + s)\cdot(y + t) = x\cdot y + \underset{\substack{\cap \\ I}}{x\cdot t} + \underset{\substack{\cap \\ I}}{s\cdot y} + \underset{\substack{\cap \\ I}}{s\cdot t}$

$= x\cdot y + r$ for $r \in I$

so the multiplication is well defined.
Check associativity & distributivity! $\rightarrow$

For example. let $x, y, z \in R$.

$$(x+I)\cdot((y+I)+(z+I)) = (x+I)\cdot((y+z)+I)$$
$$= (x\cdot(y+z)+I)$$
$$= ((x\cdot y + x\cdot z)+I) \quad \text{since } * \text{ on } R \text{ distributive}$$
$$= ((x\cdot y)+I)+((x\cdot z)+I)$$
$$= (x+I)\cdot(y+I)+(x+I)\cdot(z+I) \checkmark$$

With the operations $(x+I)+(y+I) = (x+y)+I$ and $(x+I)\cdot(y+I) = (x\cdot y)+I$, R/I is a ring called the <u>quotient</u> (or factor) ring of R by I.

<u>Thm</u>: Let $I \subseteq R$ be an ideal. Then $\exists$ a ring S and a ring homomorphism $\varphi: R \to S$ with $\ker(\varphi) = I$.
Conversely, let $\varphi: R \to S$ be a ring homomorphism, then $\ker \varphi$ is an ideal of R.