3/25/11

HW Clarification

- Let G be an abelian group. The trivial ring structure on G is given by, $\forall\, g, h \in G$, $g \cdot h = e_G$. Can check associativity & distributivity.

- A bit on $C_0(\mathbb{R})$.
observe that $C_0(\mathbb{R})$ is a vector space over the real numbers.
let $f, g \in C_0(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$

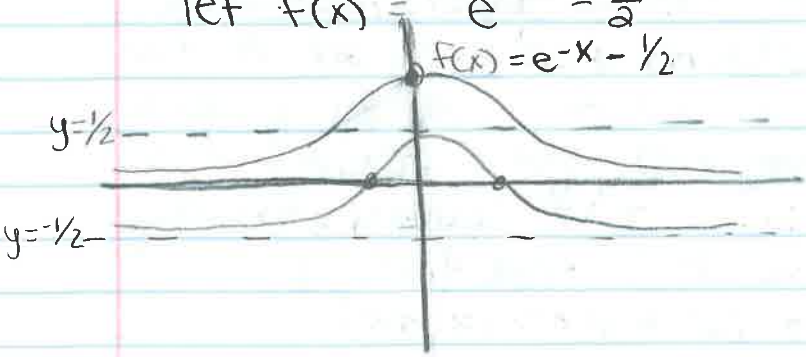
$$\lim_{|x|\to\infty} (\alpha f + \beta g)(x) = \lim_{|x|\to\infty} ((\alpha f)(x) + (\beta g)(x))$$

$$= \lim_{|x|\to\infty} (\alpha f(x) + \beta g(x)) = \alpha \lim_{|x|\to\infty} f(x) + \beta \lim_{|x|\to\infty} g(x)$$

$= 0$ so, $\alpha f + \beta g \in C_0(\mathbb{R})$

What is the dimension of $C_0(\mathbb{R})$. i.e. how big is it?

let $f(x) = e^{-x^2} - \frac{1}{2}$

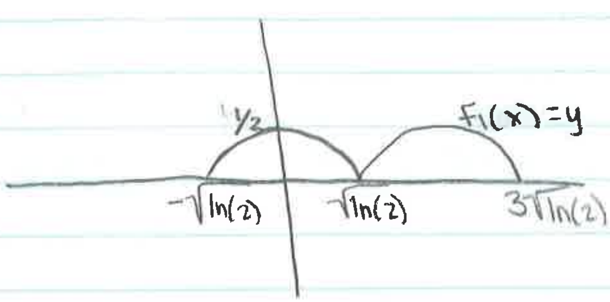


$f(x)$ crosses the x-axis at $e^{-x^2} = 1/2$

$$-x^2 = \ln(\tfrac{1}{2})$$
$$= -\ln(2)$$
$$x = \pm\sqrt{\ln(2)}$$

let $f_0(x) = \begin{cases} f(x), & |x| \leq \sqrt{\ln(2)} \\ 0, & |x| > \sqrt{\ln(2)} \end{cases}$

$f_0(x) = y$ $\qquad$ $f_0 \in C_0(x)$



Define, for $n \in \mathbb{Z}$, $f_n(x) = f_0(x - 2n\sqrt{\ln(2)})$

claim: $\{f_n\}_{n\in\mathbb{Z}}$ is a linearly independent subset of $C_0(\mathbb{R})$ [also of $C_{00}(\mathbb{R})$]

If $n_1, \dots, n_k \in \mathbb{Z}$ and $n \notin \{n_1, \dots, n_k\}$
Then $f_n(x) \neq \sum_{i=1}^{k} a_i f_{n_i}(x)$
Since $\frac{1}{2} = f_n(2n\sqrt{\ln(2)})$ but $\sum_{i=1}^{n} a_i f_{n_i}(2n\sqrt{\ln(2)}) =$ by definition. This shows $C_0(\mathbb{R})$ is infinite dimensional as a real vector space.

We could define for a fixed $x \in \mathbb{R}$, the homomorphism $\varphi_x : C_0(\mathbb{R}) \to \mathbb{R}$ by $\varphi_x(f) = f(x)$
we checked that $\varphi_x(f+g) = \varphi_x(f) + \varphi_x(g)$
Similarly, $\varphi_x(f \cdot g) = (f \cdot g)(x)$
$= f(x) \cdot f(g) = \varphi_x(f) \cdot \varphi_x(g)$
This implies that $\{\varphi_x\}_{x\in\mathbb{R}}$ are all ring homomorphisms from $C_0(\mathbb{R})$ to $\mathbb{R}$. So, $\mathrm{Ker}(\varphi_x)$ is an ideal $\forall x$.

$$\mathrm{Ker}(\varphi_x) = \{f \in C_0(\mathbb{R}) : \varphi_x(f) = 0\}$$
$$= \{f \in C_0(\mathbb{R}) : f(x) = 0\}$$

HW: show this is a maximal ideal.

3/25/11

HW Clarification

- Let G be an abelian group. The trivial ring structure on G is given by, $\forall\ g, h \in G$, $g \cdot h = e_G$. Can check associativity & distributivity.

- A bit on $C_0(\mathbb{R})$.
observe that $C_0(\mathbb{R})$ is a vector space over the real numbers.
let $f, g \in C_0(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$

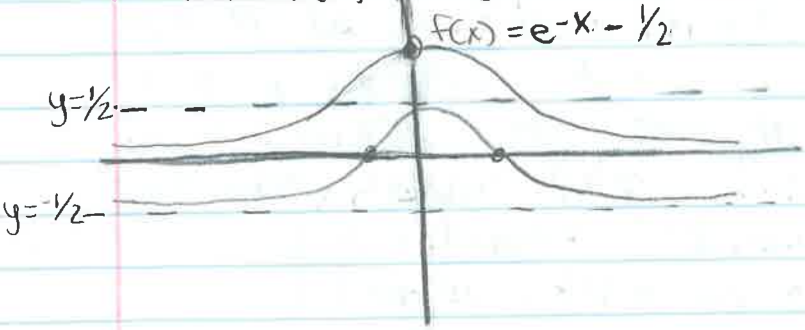
$$\lim_{|x|\to\infty} (\alpha f + \beta g)(x) = \lim_{|x|\to\infty} ((\alpha f)(x) + (\beta g)(x))$$

$$= \lim_{|x|\to\infty} (\alpha f(x) + \beta g(x)) = \alpha \lim_{|x|\to\infty} f(x) + \beta \lim_{|x|\to\infty} g(x)$$

$$= 0 \quad \text{so, } \alpha f + \beta g \in C_0(\mathbb{R})$$

What is the dimension of $C_0(\mathbb{R})$. i.e. how big is it?

let $f(x) = e^{-x^2} - \frac{1}{2}$

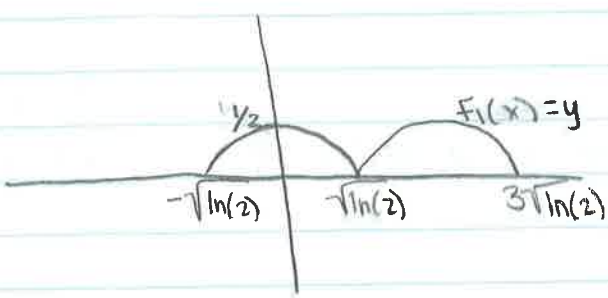


$f(x)$ crosses the x-axis at $e^{-x^2} = 1/2$

$$-x^2 = \ln(\tfrac{1}{2})$$
$$= -\ln(2)$$
$$x = \pm\sqrt{\ln(2)}$$

let $f_0(x) = \begin{cases} f(x), & |x| \le \sqrt{\ln(2)} \\ 0, & |x| > \sqrt{\ln(2)} \end{cases}$

$f_0(x) = y$ $\qquad f_0 \in C_0(x)$



Recall:

Thm: let R be a ring, $I \subseteq R$ an ideal of R. Then $\exists$ a ring S and a ring homomorphism $\varphi: R \to S$ with $\ker \varphi = I$.
Conversely: If R and S are rings and $\varphi: R \to S$ is a ring homomorphism, then $\ker \varphi$ is an ideal of R.

Pf:

$\Rightarrow$ $S = R/I$ (factor ring)
$= \{x + I : x \in R\}$
$\varphi: R \to S$
$\varphi(x) = x + I \quad \forall x \in R$ (homomorphism by def)
$\ker \varphi = \{x \in R : \varphi(x) = I\}$
$= \{x \in R : \varphi(x) = 0_R + I\}$
Suppose $x \in I$, then $\varphi(x) = x + I = I$, since I is a subring of R.
Hence $I \subseteq \ker \varphi$. now let $x \in \ker \varphi$ then $\varphi(x) = I = x + I \Rightarrow x \in I$
($x \in x + I$ and cosets are disjoint or equal) ✓

$\Leftarrow$ Suppose $\varphi: R \to S$ is a ring homomorphism.
$\ker \varphi = \{x \in R : \varphi(x) = 0_S\}$
we know from group theoretic calculations that $\ker \varphi \leq \langle S, + \rangle$. Check that $\forall y \in R$, $x \in \ker \varphi$, $xy \in \ker \varphi$. (establishes $\ker \varphi$ is a subring by setting $y \in \ker \varphi$)
$\varphi(xy) = \varphi(x)\varphi(y) = 0_S \cdot \varphi(y) = 0_S$
Similarly: $\varphi(yx) = \varphi(y)\varphi(x) = \varphi(y) 0_S = 0_S$
hence $\ker \varphi$ is an ideal of R ∎

TYPES OF RINGS

Throughout, R is a ring
and $0 \in R$ is the additive identity.

Def: $0 \neq x \in R$. Then x is called a zero divisor (or diviser of zero) if $\exists\ 0 \neq y \in R$, $xy = 0$

Examples of zero divisors:

1) $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$
$x = 2 \qquad y = 3$
$2 \cdot 3 = 6 \equiv 0 \bmod 6$, so $x = 2$, $y = 3$ are zero divisors of $\mathbb{Z}_6$

Similarly, $\mathbb{Z}_n$ has zero divisors if n isn't prime.

2) $R = M_2(\mathbb{R})$
$x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$
$xy = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Similarly, $M_n(\mathbb{R})$ has zero divisors $\forall n \geq 2$.