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Last time:

$|G| < \infty$, $H \leq G$

$[G:H] = \#$ of left cosets

left cosets:

$|gH| = |H|$

$G = \bigsqcup_{i=1}^{n} g_i H$

$|G| = n|H| \Rightarrow n = [G:H]$

Recall:

$H \trianglelefteq G$, then a group structure on the left cosets of H in G is given by $(g, k \in G)$

$gH * kH = (gk)H$

With this binary operation, the left cosets form a group.

Thm: (First Isomorphism Theorem or "Fundamental Homomorphism Theorem")

- let G, H be groups and $\varphi: G \to H$ be a homomorphism. Then $\varphi(G) \cong \{$left cosets of $Ker(\varphi)\}$

(Notation for left cosets: $G/Ker\,\varphi$)

Pf: $\varphi(G)$ is a subgroup of H since if $g \in G$,

$\varphi(g)^{-1} = \varphi(g^{-1}) \in \varphi(G)$ and if $k \in G$,

$\varphi(g)\varphi(k) = \varphi(gk) \in \varphi(G)$ (two step subgroup test)

$\varphi(G) \neq \emptyset$ since $\varphi(e_G) = e_H \in \varphi(G)$

Construct an isomorphism btw $G/Ker\,\varphi$ and $\varphi(G)$

Define $\psi: G/Ker\,\varphi \to \varphi(G)$ as for $g \in G$

$\psi(gKer\,\varphi) = \varphi(g)$, ψ is surjective by definition

check:
1) ψ is injective
2) ψ is a homomorphism
3) ψ is well-defined

(if $gKer\,\varphi = tKer\,\varphi$, then

$\psi(gKer\,\varphi) = \psi(tKer\,\varphi)$)

Examples

1) $G = \mathbb{Z}$, $H_n = \{ n \cdot k : k \in \mathbb{Z} \}$

$\mathbb{Z}$ abelian implies $H_n \trianglelefteq G$.

What is the isomorphism class of $\mathbb{Z}/H_n$?

We showed before the midterm that $[\mathbb{Z} : H_n] = n$

Claim: $\mathbb{Z}/H_n \cong \mathbb{Z}_n$

Define $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ by $\varphi(k) = k \pmod{n}$

you can check if $k, m \in \mathbb{Z}$, then

$(km) \bmod n = (k \bmod n \cdot m \bmod n) \bmod n$

This shows φ is a homomorphism.

$\ker\varphi = \{ k \in \mathbb{Z} \mid \varphi(k) = 0 \pmod{n} \}$

$\quad = \{ k \in \mathbb{Z} \mid k = nm \text{ for } m \in \mathbb{Z} \} = H_n$

φ is surjective since if $0 \le k \le n-1$, $\varphi(k) = k$

By the first isomorphism thm, $\mathbb{Z}/H_n \cong \mathbb{Z}_n$

(sometimes people write $H_n = n\mathbb{Z}$, then the isomorphism is $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$)

2) $G = S_n$, $H = A_n = \{ \sigma \in S_n : \sigma \text{ is even} \}$

Claim: $S_n/A_n \cong \mathbb{Z}_2$

Define $\varphi_n : S_n \to \mathbb{Z}_2$, $\varphi_n(\sigma) = \begin{cases} 0, & \sigma \text{ even} \\ 1, & \sigma \text{ odd} \end{cases}$

φ_n is surjective and $\ker(\varphi_n) = A_n$.

φ_n is a homomorphism since, even · even = even, odd · even = odd, and odd · odd = even.

For ex: If $\sigma_1, \sigma_2 \in S_n$ and σ_1 is odd, σ_2 is odd,

$\varphi(\sigma_1) + \varphi(\sigma_2) = 1 + 1 = 0 \bmod 2$

$\sigma_1\sigma_2$ is even, so $\varphi(\sigma_1\sigma_2) = 0$

By the first isomorphism thm, $S_n/A_n \cong \mathbb{Z}_2$

(Corollary: $|A_n| = \frac{n!}{2}$)

3) $G = SL_n(\mathbb{Z})$ for n even

$H = \{I_n, -I_n\}$

$H \trianglelefteq G$ since $H = Z(G)$

G/H is called $PSL_n(\mathbb{Z})$, the projected special linear group