4/1/11

Last Question on HW #6

$I_x \subseteq C_0(\mathbb{R})$

$I_x = \{f \in C_0(\mathbb{R}) : f(x) = 0\}$ for some fixed $x \in \mathbb{R}$

Prove: $I_x$ is maximal in $C_0(\mathbb{R})$

Question: When is $f + I_x = g + I_x$ for $f, g \in C_0(\mathbb{R})$?

$f + I_x = g + I_x$ iff $f - g \in I_x$

$\iff (f-g)(x) = 0 \iff f(x) = g(x)$

cosets are determined by the value of a function at $x$.

Suppose $\exists\ I_x \subsetneq J \subseteq C_0(\mathbb{R})$, $J$ is an ideal of $C_0(\mathbb{R})$

If $f \in J$, then $f + I_x \subseteq J$ because $J$ is a subring and contains $I_x$

Goal: for each $\alpha \in \mathbb{R}$, find a function $g_\alpha \in J \ni$ $g_\alpha(x) = \alpha$.

$$\Rightarrow \underbrace{g_\alpha + I_x}_{\{f \in C_0(\mathbb{R}) : f(x) = \alpha\}} \subseteq J$$

$$\bigsqcup g_\alpha + I_x = C_0(\mathbb{R})$$

$$\Rightarrow J = C_0(\mathbb{R})$$

Achieve the goal by constructing $g_\alpha$. Note that $I_x \subsetneq J \Rightarrow \exists\ g \in J,\ g(x) \neq 0$

Define

$$g_\alpha(y) = \frac{\alpha g^2(y)}{g^2(x)}$$

Note: $g_\alpha(x) = \alpha$ and $g_\alpha \in C_0(\mathbb{R})$.

$g_\alpha \in J$ since $g_\alpha = \underset{\in J}{g} \cdot \underset{\in C_0}{\left(\frac{\alpha g}{g^2(x)}\right)}$

and $J$ an ideal, so $g_\alpha \in J$ ✓

$R/I \cong \mathbb{C}$ or $\mathbb{R}$ or ground field $\Rightarrow I$ maximal

Last Section:

## FIELD THEORY

From now on, $F$ will be a field (commutative ring with no zero divisors $\ni$ every nonzero element in $F$ is a unit)

Additive identity of $F$: $0$

multiplicative unit of $F$: $1$

Examples

1) We've already proved that $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$ ($p$ prime) are fields

2) field of order four

$F = \{0, 1, x, y\}$

Addition: $x + x = y + y = 0$

$x + y = 1$

$1 + 1 = 0$

($x + x = y + y = 1 + 1 = 0$

$\Rightarrow x = -x, \ y = -y, \ 1 = -1$

so $x = 1 - y = 1 + y$ for example)

| + | x | y | 1 |
|---|---|---|---|
| x | 0 | 1 | y |
| y | 1 | 0 | x |
| 1 | y | x | 0 |

| • | x | y |
|---|---|---|
| x | y | 1 |
| y | 1 | x |

commutative, every nonzero element is a unit

Multiplication:

Check if it is a ring -

$(F, +) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Associativity: (of mult)

$y \cdot (xy) = y \cdot 1 = y$ $\quad$ $x \cdot (yx) = x \cdot 1 = x$

$(xy) \cdot y = 1 \cdot y = y$ ✓ $\quad$ $(xy) \cdot x = 1 \cdot x = x$ ✓

sufficient.

Distributivity:

$x \cdot (x+y) = x \cdot 1 = x$ $\quad$ $x \cdot x + x \cdot y = y + 1 = x$ ✓

$$\left.\begin{array}{l} x \cdot (x+1) \\ x \cdot (x+x) \\ x \cdot (y+y) \\ x \cdot (1+1) \\ x \cdot (y+1) \end{array}\right\} \text{Check rest on HW!}$$

Note: If F and E are fields, then $F \times E$ is a ring, but not a field!

Reason: If $0_E$ is the additive identity of E and $0_F$ is the additive identity of F, then if $x \in F$, $x \neq 0_F$, and $y \in E$, $y \neq 0_E$, then $(x, 0_F)(0_E, y) = (0_E, 0_F)$

Hence $(x, 0_F)$ + $(0_E, y)$ are zero divisors

Prop: Let F be a field, The only ideals of F are $\{0\}$ and F

Pf: like the case $F = \mathbb{Q}$ ...

Let I be an ideal of F, $I \neq \{0\}$. Then $\exists$ $x \in I$, $x \neq 0$. Since F is a field, x is a unit. Denoting the inverse by $x^{-1}$, $x^{-1} \cdot x = 1 \in I$ since $x \in I$ and I is an ideal. by HW, $1 \in I$

$\Rightarrow I = F$ ∎

**Def:** Let $K \subseteq F$. Then $K$ is a subfield of $F$ if $K$ is a field with the same operations as $F$.

**Thm:** (Subfield Test) Suppose $K \subseteq F$, $K \neq \emptyset$. Then $K$ is a subfield of $F$ iff $\forall x, y \in K$, $x - y \in K$ and $\forall$ units $x, y \in K^{\times}$, $xy^{-1} \in K$.
**Pf:** Two applications of the subgroup test $\blacksquare$

Since ideals are scarce, we look outside the field instead of inside.

**Def:** A field $E$ is called an extension of $F$, if $F$ is a subfield of $E$.

Examples:

1) If $F = \mathbb{Q}$, $E = \mathbb{R}$ or $E = \mathbb{C}$ are field extensions.
2) If $F = \mathbb{R}$, $E = \mathbb{C}$ is a field extension.
3) if $F = \mathbb{Z}_3$, $E = \mathbb{Z}_3[i] = \{a + bi : a, b \in \mathbb{Z}_3\}$

Why a field?

$(a+bi)(c+di)$ reduce coefficients mod p.

$= ac + adi + bci - bd$

$= \underbrace{(ac - bd)}_{\text{reduce mod } p} + \underbrace{(ad + bc)}\, i$

$(a+bi)^{-1} = \dfrac{1}{a+bi} = \dfrac{a-bi}{a^2+b^2}$ a & b not both zero