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**Def** An abelian group $V$ is called a vector space over $F$ if $\exists$ a map $\varphi: F \times V \to V$ satisfying $\forall\ \alpha, \beta \in \mathbb{F}$ and $v, w \in V$

1) $\varphi(1, v) = v$
2) $\varphi(\alpha, v+w) = \varphi(\alpha, v) + \varphi(\alpha, w)$
3) $\varphi(\alpha + \beta, v) = \varphi(\alpha, v) + \varphi(\beta, v)$
4) $\varphi(\alpha\beta, v) = \varphi(\alpha, \beta v)$

In reality we supress $\varphi$ and write $\varphi(\alpha, v) = \alpha V$

The rules become:

1) $1 \cdot V = V$
2) $\alpha \cdot (v + w) = \alpha \cdot v + \alpha \cdot w$
3) $(\alpha + \beta) \cdot v = \alpha v + \beta v$
4) $(\alpha\beta) \cdot v = \alpha(\beta v)$

Examples:

1) $F^n := \underbrace{F \times F \times \cdots \times F}_{n\text{ times}}$ for $n \in \mathbb{N}$ is a vector space over $F$. $V = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in F^n$ for $\alpha_i \in F$, $1 \le i \le n$. If $\alpha \in F$, $\alpha \cdot V = (\alpha\alpha_1, \alpha\alpha_2, \ldots, \alpha\alpha_n)$

2) $C_0(\mathbb{R})$ is a vector space over $F = \mathbb{R}$. If $f \in C_0(\mathbb{R})$ $v = f$ and $\alpha \in \mathbb{R}$, $(\alpha \cdot f)(x) = \alpha f(x)$ $\forall\ x \in \mathbb{R}$.

3) $V = F[x] = \{$polynomials with coefficients in $F\}$ If $v = p(x) = \sum_{i=0}^{n} a_i x^i$ for $\alpha_i \in F$, $1 \le i \le n$, then $\alpha \cdot v = \sum_{i=0}^{n} (\alpha a_i) x^i$. vector space over $F$.

4) If $E$ is any extension field of $F$, then $V = E$ is a vector space over $F$. If $\alpha \in F$, $v \in E$, since $F \subseteq E$, we can just take the multiplication $\alpha \cdot v \in E$ to be the map $\varphi$.

e.g. $F = \mathbb{R}$, $E = \mathbb{C}$, $\alpha \in \mathbb{R}$, $v = a+bi \in \mathbb{C}$
for $a, b \in \mathbb{R}$
$\alpha(a+bi) = \alpha a + \alpha bi$ since we know
how to multiply two complex numbers & $\mathbb{R} \in \mathbb{C}$

**Def:** $V$ is a vector space over $F$. A subset $S \subseteq V$ is called <u>linearly independent</u> if $\forall$ $v_1, v_2, \dots v_n \in S$ and $\alpha_1, \alpha_2, \dots \alpha_n \in F$,
$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \Rightarrow$
$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

<u>Example</u>

In $F^n$, the set $\{e_i\}_{i=1}^n$, where
$e_i = (0, \dots, 0, 1, 0, \dots 0)$
$\uparrow$ $i$th position

or equivalently, $e_i = (\alpha_1, \dots, \alpha_n)$ where $\alpha_i = 1$ and $\alpha_j = 0$ $\forall$ $j \neq i$

**Def:** The <u>dimension</u> of a vector space $V$ over $F$ is the cardinality of a maximal (in the sense of set inclusion) linearly independent set. Such a set is called a (Hamel) basis.

<u>Note:</u> A basis for $V$ over $F$ always exists (Axiom of Choice)

<u>Examples:</u>

1) In $F^n$, the set $\{e_i\}_{i=1}^n$ is a basis, so the dimension of $F^n$ is $n$.

2) In $F[x]$, a basis is given by $\{x^i\}_{i=0}^{\infty}$
The dimension of $F[x]$ is infinite, with cardinality that of the natural numbers.

3) $C_0(\mathbb{R})$ has infinite dimension over $\mathbb{R}$ (we proved this in class) The dimension has uncountable cardinality.

Def: If $E$ is an extension field of $F$, the degree of $E$ is the dimension of $E$ as a vector space over $F$.

Examples

1) $\mathbb{Q} = F$
$E = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$
basis is $\{1, \sqrt{2}\}$, so the degree of $E$ is 2.

2) $F = \mathbb{Q}$
$E = \mathbb{Q}[\sqrt[3]{7}] = \{a + b\sqrt[3]{7} + c(\sqrt[3]{7})^2 : a, b, c \in \mathbb{Q}\}$
basis is $\{1, \sqrt[3]{7}, (\sqrt[3]{7})^2\}$, degree of $E$ is 3.

3) $F = \mathbb{Q}$
$E = \mathbb{Q}[\pi]$ = the smallest subfield of $\mathbb{R}$ containing both $\mathbb{Q}$ and $\pi$.
degree of $E$ = infinity, what is cardinality?

Def: $E$ is an extension field of $F$. Then $t \in E$ is called algebraic over $F$ if $t$ is a zero of a nonconstant polynomial in $F[x]$. If $t$ is not algebraic over $F$, we say $t$ is transcedental over $F$