

Definition: (order) The **order** of a

cycle f in S_n is the

cardinality of the T_1

block, i.e., the cardinality

of the set of all $m \in \{1, 2, \dots, n\}$

such that $f(m) \neq m$.

If g is the product (i.e.

composition) of two **disjoint**

cycles, $g = g_1 \circ g_2$ where

g_1 and g_2 are disjoint cycles,

we define the order of g

to be the least common multiple
(lcm) of the orders of g_1 and g_2 .

Example 5: (orders) If f is the cycle given by $(1\ 2\ 6\ 8)$,

then the order of f is

4. If g is the

permutation given by

$$g = (2\ 3\ 5)(4\ 9),$$

the order of g is

$$3 \cdot 2 = 6.$$

But if

$$h = (1\ 2\ 3\ 5)(4\ 9),$$

the order of h is 4

since the lcm of 4 and

2 is 4.

Theorem: (disjoint cycle decomposition)

Let $f \in S_n$. Then \exists disjoint

cycles f_1, f_2, \dots, f_m for $m \in \mathbb{N}$

such that

$$f = f_1 \circ f_2 \circ \dots \circ f_m$$

proof: We proceed by induction on n .

For $n=1$ or $n=2$, the

result is trivial, regarding

(n) as a cycle of order 1

for $m \in \{1, 2, \dots, n\}$.

In S_3 , for example,

$$e = (1)(2)(3)$$

To establish the result in S_3 ,

we have 6 permutations:

$$(123) \quad (\text{cycle})$$

$$(132) \quad (\text{cycle})$$

$$(12) = (12)(3) \quad (\text{cycle})$$

$$(13) = (13)(2) \quad (\text{cycle})$$

$$(23) = (23)(1) \quad (\text{cycle})$$

In S_4 , we finally see something that is neither a cycle nor the identity:

$$(1\ 2)(3\ 4) \rightarrow \text{product of 2 disjoint cycles of order 2.}$$

$$(1\ 3)(2\ 4)$$

$$(1\ 4)(2\ 3)$$

are all elements that are products of 2 disjoint cycles of order 2.

Suppose the result is true for

S_k for $k \leq n$. Prove the result

for S_{n+1} . Take $f \in S_{n+1}$.

If f is a cycle, then we are

done. If f is not a cycle,

consider successive powers

$f^{(l)}(n+1)$. If $f(n+1) = n+1$,

Then $f|_{\{1, 2, \dots, n\}}$ (notation means

" f restricted to the domain $\{1, 2, \dots, n\}$)

defines a permutation $g \in S_n$.

By induction, g is the product of disjoint cycles, and since $f = g(n+1)$, f is also a product of disjoint cycles.

Now suppose $f(n+1) \neq n+1$.

Then since f is not a cycle, $\exists l \in \mathbb{N}$, $1 < l < n+1$, with $f^{(l)}(n+1) = n+1$.

Let t be the minimal power of f with $f^{(t)}(n+1) = n+1$.

Let $\{x_1, x_2, \dots, x_t\}$ denote

the powers of f applied to $n+1$,

so $x_i = f^{(i)}(n+1) \wedge$

$1 \leq i \leq t$. Let

$$A = S_{n+1} \setminus \{x_1, x_2, \dots, x_t\}$$

(here, $S_{n+1} \setminus \{x_1, x_2, \dots, x_t\}$

is the relative complement of

$\{x_1, x_2, \dots, x_t\}$ in S_{n+1} :

$$S_{n+1} \setminus \{x_1, x_2, \dots, x_t\} = S_{n+1} \cap \{x_1, x_2, \dots, x_t\}^c$$

Let $h = f|_A$. Since f is

not a cycle, $|A| < n+1$.

By induction, h may be expressed

as a product of disjoint cycles.

Note: here we extended our definition

of S_n to include all

sets of n elements -

This is to cover cases like

$(f \in S_6)$

$$f = (1\ 2\ 5)\ (3\ 4\ 6)$$

$\{1, 2, 5\} \neq \{1, 2, 3\}$, but

they have the same cardinality,

so the disjoint cycle statement

should apply to bijections on

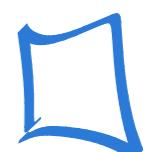
$\{1, 2, 5\}$ as well.

Now we are done, since

$$f = h(x_1, x_2, \dots, x_t)$$

and h is a product of

disjoint cycles.



Example 6 : (applying the theorem)

Write , for

$$f = (1 \ 2 \ 3)(1 \ 2 \ 4 \ 5)(4 \ 5 \ 6) \in S_7,$$

as a product of disjoint cycles.

Solution: Multiplying out using cycle notation .

$$f = (1 \ 3)(2 \ 4)(5 \ 6)(7) \checkmark$$

Observations About S_n

- 1) S_n has a binary operation, function composition : if $f, g \in S_n$,
 $f \circ g \in S_n$
- 2) S_n has a "neutral" element
with respect to composition, the
element e : $e \circ f = f \circ e = f$
 $\forall f \in S_n$.
- 3) Since every $f \in S_n$ is a bijection,
 f has an "inverse" f^{-1} such
that $f \circ f^{-1} = f^{-1} \circ f = e$.

4) Function composition is
associative, but **not**
commutative!