Definition: (order) The order of a cycle f in S_n is the cardinality of the T_1 block, i.e., the cardinality of the set of all $m \in \{1, 2, \dots, n\}$ such that $f(m) \neq m$.

If g is the product (i.e. composition) of two disjoint cycles, $g = g_1 \circ g_2$ where g_1 and g_2 are disjoint cycles, we define the order of g to be the least common multiple (lcm) of the orders of g_1 and g_2.

Example 5: (orders) If f is the cycle given by $(1\ 2\ 6\ 8)$, then the order of f is 4. If g is the permutation given by

$$g = (2\ 3\ 5)(4\ 9),$$

the order of g is

$$3 \cdot 2 = 6.$$

But if

$$h = (1\ 2\ 3\ 5)(4\ 9),$$

the order of h is 4

since the lcm of 4 and

2 is 4.

Theorem: (disjoint cycle decomposition)

Let $f \in S_n$. Then $\exists$ disjoint cycles $f_1, f_2, \ldots, f_m$ for $m \in \mathbb{N}$ such that

$$\boxed{f = f_1 \circ f_2 \circ \ldots \circ f_m}$$

proof: We proceed by induction on n.

For $n=1$ or $n=2$, the result is trivial, regarding (m) as a cycle of order 1 for $m \in \{1, 2, \ldots, n\}$.

In S_3, for example,

$$e = (1)(2)(3)$$

To establish the result in S_3,

we have 6 permutations:

$(1\,2\,3)$ (cycle)

$(1\,3\,2)$ (cycle)

$(1\,2) = (1\,2)(3)$ (cycle)

$(1\,3) = (1\,3)(2)$ (cycle)

$(2\,3) = (2\,3)(1)$ (cycle)

In S_4, we finally see something that is neither a cycle nor the identity:

$(1\ 2)(3\ 4)$ → product of 2 disjoint cycles of order 2.

$(1\ 3)(2\ 4)$

$(1\ 4)(2\ 3)$

are all elements that are products of 2 disjoint cycles of order 2.

Suppose the result is true for S_k for $k \leq n$. Prove the result for S_{n+1}. Take $f \in S_{n+1}$.

If f is a cycle, then we are done. If f is not a cycle, consider successive powers $f^{(l)}(n+1)$. If $f(n+1) = n+1$, Then $f\big|_{\{1,2,\ldots,n\}}$ (notation means "f restricted to the domain $\{1,2,\ldots,n\}$") defines a permutation $g \in S_n$.

By induction, g is the product of disjoint cycles, and since $f = g\,(n+1)$, f is also a product of disjoint cycles.

Now suppose $f(n+1) \neq n+1$.

Then since f is not a cycle, $\exists\ l \in \mathbb{N}$, $1 < l < n+1$, with $f^{(l)}(n+1) = n+1$.

Let t be the minimal power of f with $f^{(t)}(n+1) = n+1$.

Let $\{x_1, x_2, \ldots, x_t\}$ denote

the powers of f applied to $n+1$,

so $x_i = f^{(i)}(n+1)$ $\forall$

$1 \leq i \leq t$. Let

$$\boxed{A = S_{n+1} \setminus \{x_1, x_2, \ldots, x_t\}}$$

(here, $S_{n+1} \setminus \{x_1, x_2, \ldots, x_t\}$

is the relative complement of

$\{x_1, x_2, \ldots, x_t\}$ in S_{n+1}:

$S_{n+1} \setminus \{x_1, x_2, \ldots, x_t\} = S_{n+1} \cap \{x_1, x_2, \ldots, x_t\}^c$)

Let $h = f|_A$. Since f is not a cycle, $|A| < n+1$.

By induction, h may be expressed as a product of disjoint cycles.

Note: here we extended our definition of S_n to include all sets of n elements.

This is to cover cases like

$(f \in S_6)$

$f = (1\ 2\ 5)(3\ 4\ 6)$

$\{1,2,5\} \neq \{1,2,3\}$, but they have the same cardinality, so the disjoint cycle statement should apply to bijections on $\{1,2,5\}$ as well.

Now we are done, since

$$f = h\,(x_1\ x_2\ \dots\ x_t)$$

and h is a product of disjoint cycles.

$\square$

Example 6: (applying the theorem)

Write, for

$$f = (1\ 2\ 3)(1\ 2\ 4\ 5)(4\ 5\ 6) \in S_7,$$

as a product of disjoint cycles.

Solution: Multiply out using cycle notation.

$$f = (1\ 3)(2\ 4)(5\ 6)(7)$$

Observations About S_n

1) S_n has a binary operation, function composition: if $f, g \in S_n$, $f \circ g \in S_n$

2) S_n has a "neutral" element with respect to composition, the element e: $e \circ f = f \circ e = f$ $\forall f \in S_n$.

3) Since every $f \in S_n$ is a bijection, f has an "inverse" f^{-1} such that $f \circ f^{-1} = f^{-1} \circ f = e$.

4) Function composition is associative, but not commutative!