Permutations

(Section 1.5)

Recall from HW 1: the number of bijections from an n element set to itself is $n!$ — we'll denote the set of all such bijections by S_n.

Permutations = "the symmetries of a configuration of identical objects"

To keep track, we number the objects. We usually use the natural numbers to keep this in order. Any such symmetry is called a permutation.

Notice: (composition) if f and g are elements of S_n, then $f \circ g$, the composition, is also an element of S_n.

Why? By the pigeonhole principle, injectivity of $f \circ g$ will imply surjectivity, so let's check injectivity: let $k \in \{1, 2, \ldots, n\}$.

Suppose $(f \circ g)(k) = (f \circ g)(m)$ for $m \in \{1, 2, \ldots, n\}$.

We want to show $k = m$.

$(f \circ g)(k) = (f \circ g)(n)$ means

$f(g(k)) = f(g(m))$.

Since f is bijective, f is also injective, so

$g(k) = g(n)$.

Since g is bijective, g is also injective, so

$k = m$.

Example 1: (composition in S_4)

Let $f, g \in S_4$,

$f(1) = 3$	$g(1) = 2$
$f(2) = 2$	$g(2) = 1$
$f(3) = 4$	$g(3) = 3$
$f(4) = 1$	$g(4) = 4$.

Let's compute $f \circ g$ and $g \circ f$.

$$(f \circ g)(1) = f(g(1)) = f(2) = 2$$

$$(f \circ g)(2) = f(g(2)) = f(1) = 3$$

$$(f \circ g)(3) = f(g(3)) = f(3) = 4$$

$$(f \circ g)(4) = f(g(4)) = f(4) = 1$$

Then

$$(g \circ f)(1) = g(f(1)) = g(3) = 3$$

(note since $(g \circ f)(1) \neq (f \circ g)(1)$, $g \circ f \neq f \circ g$!)

$$(g \circ f)(2) = g(f(2)) = g(2) = 1$$

$$(g \circ f)(3) = g(f(3)) = g(4) = 4$$

$$(g \circ f)(4) = g(f(4)) = g(1) = 2$$

We need better notation!

Cauchy's 2-line notation:

$$f: \begin{bmatrix} 1 & 2 & 3 & 4 \\ f(1) & f(2) & f(3) & f(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

$$g: \begin{bmatrix} 1 & 2 & 3 & 4 \\ g(1) & g(2) & g(3) & g(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

To compute $f \circ g$, stack g on top of f and follow the path!

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

$(f \circ g)(1) = 2$

$(f \circ g)(2) = 3$

$(f \circ g)(3) = 4$

$(f \circ g)(4) = 1$