Proposition: (ring properties)

Let K be a field, $p(x), q(x), r(x)$ elements of $K[x]$.

1) Addition is commutative and associative.

2) Every element of $K[x]$ has an additive inverse.

3) Multiplication is commutative and associative.

4) $p(x) = 1$, the multiplicative identity of K, is a multiplicative identity for $K[x]$.

5) Multiplication distributes over addition

proof: (incomplete)

1) Let $p(x) = \sum_{i=0}^{n} a_i x^i \in K[x]$,

$$q(x) = \sum_{t=0}^{m} b_t x^t \in K[x],$$

$$r(x) = \sum_{j=0}^{\ell} c_j x^j \in K[x].$$

Temporarily assume that

$n \geq m \geq l$.

Then

$$p(x)+q(x) = \sum_{i=0}^{m} (a_i+b_i)x^i + \sum_{i=m+1}^{n} a_i x^i$$

$$= \sum_{i=0}^{n} (b_i+a_i)\, x^i + \sum_{i=m+1}^{n} a_i x^i$$

$a_i+b_i = b_i+a_i$ since

K is a field

$$= q(x)+p(x)$$

Now for associativity:

$$(p(x)+q(x))+r(x)$$

$$=\left(\sum_{i=0}^{m}(a_i+b_i)x^i+\sum_{i=m+1}^{n}a_ix^i\right)+r(x)$$

$$=\sum_{i=0}^{l}((a_i+b_i)+c_i)x^i+\sum_{i=l+1}^{m}(a_i+b_i)x^i$$

$$+\sum_{i=m+1}^{n}a_ix^i$$

Since K is a field,

$$\underline{(a_i+b_i)+c_i=a_i+(b_i+c_i)},$$

so we get

$(p(x)+q(x))+r(x)$

$$= \sum_{i=0}^{l} (a_i + (b_i + c_i)) x^i + \sum_{i=l+1}^{m} (a_i + b_i) x^i$$

$$+ \sum_{i=m+1}^{n} a_i x^i$$

$$= p(x) + (q(x) + r(x))$$

2) The additive identity of $K[x]$

is 0 = zero polynomial.

Given $p(x) = \sum_{i=0}^{n} a_i x^i$,

set $-p(x) = \sum_{i=0}^{n} (-a_i) x^i$

where $-a_i \in K$ is the additive inverse of a_i, $0 \le i \le n$.

By definition of addition,

$p(x) + (-p(x))$

$$= \sum_{i=0}^{n} (a_i + (-a_i)) x^i$$

$$= \sum_{i=0}^{n} 0 \cdot x^i = 0$$

5)

$$p(x)\cdot\left(q(x)+r(x)\right)$$

$$=p(x)\cdot\left(\sum_{t=0}^{\ell}(c_t+b_t)x^t+\sum_{t=\ell+1}^{m}b_tx^t\right)$$

$$=\sum_{i=0}^{n}\sum_{t=0}^{\ell}a_i\cdot(c_t+b_t)x^{i+t}$$

$$+\sum_{i=0}^{n}\sum_{t=\ell+1}^{m}a_ib_tx^{i+t}$$

Since K is a field,

$$a_i \cdot (c_t + b_t) = a_i \cdot c_t + a_i \cdot b_t$$

$\forall\ 0 \le i \le n,\ 0 \le t \le l.$

Substituting,

$$p(x) \cdot (q(x) + r(x))$$

$$= \sum_{i=0}^{n} \sum_{t=0}^{l} (a_i \cdot c_t + a_i \cdot b_t) x^{i+t}$$

$$+ \sum_{i=0}^{n} \sum_{t=l+1}^{m} a_i b_t x^{i+t}$$

(definition of polynomial addition)

$$= \sum_{i=0}^{n} \sum_{t=0}^{l} a_i c_t x^{i+t}$$

$$+ \sum_{i=0}^{n} \sum_{t=0}^{l} a_i b_t x^{i+t}$$

$$+ \sum_{i=0}^{n} \sum_{t=l+1}^{m} a_i b_t x^{i+t}$$

combine

$$= \sum_{i=0}^{n} \sum_{t=0}^{l} a_i c_t x^{i+t}$$

$$+ \sum_{i=0}^{n} \sum_{t=0}^{m} a_i b_t x^{i+t}$$

$$= p(x)\cdot r(x) + p(x)\cdot q(x)$$

$$= p(x)\cdot q(x) + p(x)\cdot r(x)$$

Leave the rest!