

## Proposition: (ring properties)

Let  $K$  be a field,  $p(x), q(x), r(x)$   
elements of  $K[x]$ .

- 1) Addition is commutative  
and associative.
- 2) Every element of  $K[x]$   
has an additive inverse.
- 3) Multiplication is commutative  
and associative.

4)  $p(x) = 1$ , the multiplicative identity of  $K$ , is a multiplicative identity for  $K[x]$ .

5) Multiplication distributes over addition

proof: (incomplete)

$$1) \text{ Let } p(x) = \sum_{i=0}^m a_i x^i \in K[x],$$

$$q(x) = \sum_{t=0}^n b_t x^t \in K[x],$$

$$r(x) = \sum_{j=0}^l c_j x^j \in K[x].$$

Temporarily assume that

$$n \geq m \geq l.$$

Then

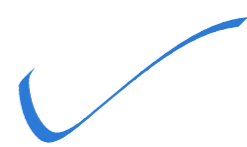
$$p(x) + q(x) = \sum_{i=0}^n (a_i + b_i) x^i + \sum_{i=m+1}^n a_i x^i$$

$$= \sum_{i=0}^n (b_i + a_i) x^i + \sum_{i=m+1}^n a_i x^i$$

$a_i + b_i = b_i + a_i$  since

$K$  is a field

$$= q(x) + p(x)$$



Now for associativity,

$$(p(x) + q(x)) + r(x)$$

$$= \left( \sum_{i=0}^m (a_i + b_i) x^i + \sum_{i=m+1}^n a_i x^i \right) + r(x)$$

$$= \sum_{i=0}^l \left( (a_i + b_i) + c_i \right) x^i + \sum_{i=l+1}^m (a_i + b_i) x^i$$

$$+ \sum_{i=m+1}^n a_i x^i$$

Since  $K$  is a field,

$$\underline{(a_i + b_i) + c_i = a_i + (b_i + c_i)},$$

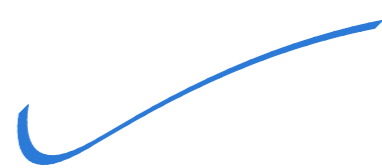
So we get

$$(p(x) + q(x)) + r(x)$$

$$= \sum_{i=0}^{\ell} (a_i + (b_i + c_i)) x^i + \sum_{i=\ell+1}^m (a_i + b_i) x^i$$

$$+ \sum_{i=m+1}^n a_i x^i$$

$$= p(x) + (q(x) + r(x))$$



2) The additive identity of  $K[x]$

is  $0 =$  zero polynomial.

$$\text{Given } p(x) = \sum_{i=0}^n a_i x^i,$$

$$\text{set } -p(x) = \sum_{i=0}^n (-a_i) x^i$$

where  $-a_i \in K$  is the additive inverse of  $a_i$ ,  $0 \leq i \leq n$ .

By definition of addition,

$$p(x) + (-p(x))$$

$$= \sum_{i=0}^n (a_i + (-a_i)) x^i$$

$$= \sum_{i=0}^n 0 \cdot x^i = 0 \quad \checkmark$$

5)

$$p(x) \cdot (q(x) + r(x))$$

$$= p(x) \cdot \left( \sum_{t=0}^{\ell} (c_t + b_t) x^t + \sum_{t=\ell+1}^m b_t x^t \right)$$

$$= \sum_{i=0}^n \sum_{t=0}^{\ell} a_i (c_t + b_t) x^{i+t}$$

$$+ \sum_{i=0}^n \sum_{t=\ell+1}^m a_i b_t x^{i+t}$$

Since  $K$  is a field,

$$\underline{a_i \cdot (c_t + b_t) = a_i \cdot c_t + a_i \cdot b_t}$$

$$\forall 0 \leq i \leq n, 0 \leq t \leq l.$$

Substituting,

$$p(x) \cdot (q(x) + r(x))$$

$$= \sum_{i=0}^n \sum_{t=0}^l (a_i \cdot c_t + a_i \cdot b_t) x^{i+t}$$

$$+ \sum_{i=0}^n \sum_{t=l+1}^m a_i b_t x^{i+t}$$



(definition of polynomial addition)

$$= \sum_{i=0}^n \sum_{t=0}^l a_i c_t x^{i+t}$$

$$+ \sum_{i=0}^n \sum_{t=0}^l a_i b_t x^{i+t}$$

$$+ \sum_{i=0}^n \sum_{t=l+1}^n a_i b_t x^{i+t}$$

combine

$$= \sum_{i=0}^n \sum_{t=0}^l a_i c_t x^{i+t}$$

$$+ \sum_{i=0}^n \sum_{t=0}^m a_i b_t x^{i+t}$$

$$= p(x) \cdot r(x) + p(x) \cdot q(x)$$

$$= p(x) \cdot q(x) + p(x) \cdot r(x) \quad \checkmark$$

Leave the rest!