Rings and Fields

(Section 1.11)

Some special examples: We know that either $\mathbb{Z}$ or $\mathbb{Z}_n$ for $n \in \mathbb{N}$, $n \geq 2$, have two binary operations, one of which gives a (commutative) group structure, and the other is associative and distributes over the group operation. These are the ingredients for a ring!

Definition: (ring) A ring is a set R endowed with two binary operations, "$+$" and "$\cdot$", such that $\forall\, x, y, z \in R$,

1) $(R, +)$ is a group

2) $x + y = y + x$ ("$+$" is commutative)

3) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
(associativity of "$\cdot$")

4) $x \cdot (y + z) = x \cdot y + x \cdot z$
$(x + y) \cdot z = x \cdot z + y \cdot z$
(distributivity of "$\cdot$" over "$+$")

Example 1: ($M_2(\mathbb{R})$) Let

$M_2(\mathbb{R})$ denote the set of all 2×2 matrices with real entries.

Let $A = [a_{i,j}]_{i,j=1}^{2}$,

$B = [b_{i,j}]_{i,j=1}^{2}$, and

$C = [c_{i,j}]_{i,j=1}^{2}$ be

in $M_2(\mathbb{R})$.

Define

$$\boxed{A+B = [a_{i,j} + b_{i,j}]_{i,j=1}^{2}}$$

(add the corresponding entries)

$A \cdot B$ is the 2×2 matrix

with

$$\boxed{(A \cdot B)_{i,j} = (a_{i,1} b_{1,j} + a_{i,2} b_{2,j})}$$

$\forall \; 1 \leq i,j \leq 2$.

Under these operations, $M_2(\mathbb{R})$ is a ring!

Check! Since $(\mathbb{R}, +)$ is a commutative group and addition is component-wise, $(M_2(\mathbb{R}), +)$ is a commutative group.

Let's check distributivity in one direction:

$$A(B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}(b_{11}+c_{11}) + a_{12}(b_{21}+c_{21})) & (a_{11}(b_{12}+c_{12}) + a_{12}(b_{22}+c_{22})) \\ (a_{21}(b_{11}+c_{11}) + a_{22}(b_{21}+c_{21})) & (a_{21}(b_{12}+c_{12}) + a_{22}(b_{22}+c_{22})) \end{bmatrix}$$

multiplication distributes over addition in $\mathbb{R}$

$$= \begin{bmatrix} (a_{11}b_{11} + a_{11}c_{11} + a_{12}b_{21} + a_{12}c_{21}) & (a_{11}b_{12} + a_{11}c_{12} + a_{12}b_{22} + a_{12}c_{22}) \\ (a_{21}b_{11} + a_{21}c_{11} + a_{22}b_{21} + a_{22}c_{21}) & (a_{21}b_{12} + a_{21}c_{12} + a_{22}b_{22} + a_{22}c_{22}) \end{bmatrix}$$

definition of "+" in $M_2(\mathbb{R})$

$$= \begin{bmatrix} (a_{11} b_{11} + a_{12} b_{21}) & (a_{11} b_{12} + a_{12} b_{22}) \\ (a_{21} b_{11} + a_{22} b_{21}) & (a_{21} b_{12} + a_{22} b_{22}) \end{bmatrix}$$

$$+ \begin{bmatrix} (a_{11} c_{11} + a_{12} c_{21}) & (a_{11} c_{12} + a_{12} c_{22}) \\ (a_{21} c_{11} + a_{22} c_{21}) & (a_{21} c_{12} + a_{22} c_{22}) \end{bmatrix}$$

$$= A \cdot B + A \cdot C$$