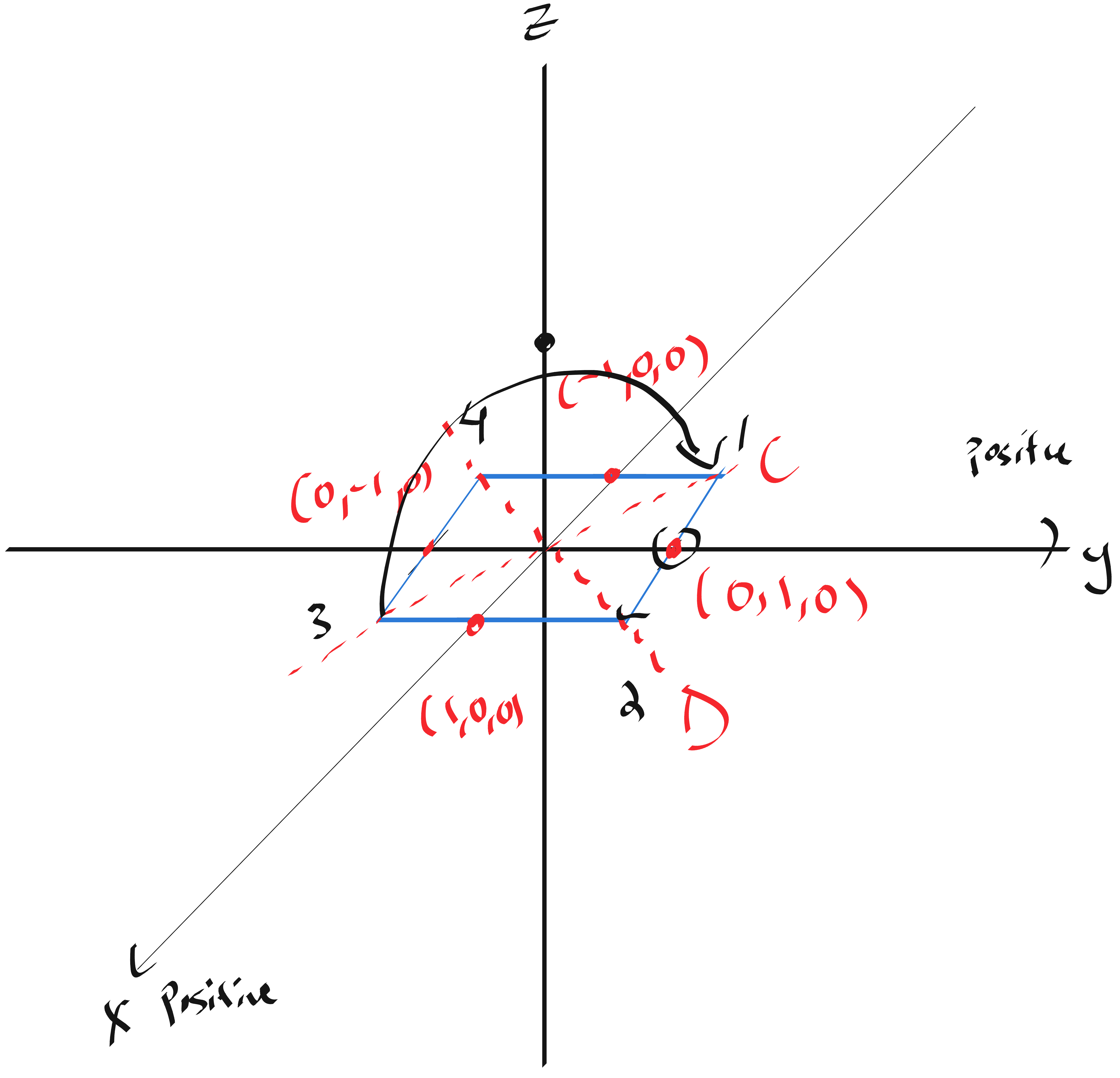
# Symmetries of the Square as Matrices

First: fix the square in the xy-plane



Matrix for $r$: (clockwise rotation by $90^\circ$)

We just need to know where $e_1$, $e_2$, and $e_3$ go to under $r$.

$$r(e_3) = r\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$r(e_2) = r\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r(e_1) = r\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Matrix: $$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix for $A$:

$$A(e_3) = A\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A(e_2) = A\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$A(e_1) = A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

## Matrix for B :

$$B(e_3) = B\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$B(e_2) = B\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$B(e_1) = B\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

We get

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Matrix for C :

$$C(e_3) = C\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$C(e_2) = C\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$C(e_1) = C\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

We get

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

## Matrix for D:

$$D(e_3) = D\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$D(e_2) = D\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D(e_1) = D\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We get

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Recall $r$ is given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r^2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = r^{-1} \checkmark$$

$$r^4 = e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q: Are there more symmetries of the square?

A: No, but why?

Note that any symmetry must take vertices to vertices and edges to edges. Since symmetries are isometries, midpoints of edges go to midpoints of edges.

Denote a symmetry of the square by $f$. Midpoints are $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}-1\\0\\0\end{bmatrix}$, and $\begin{bmatrix}0\\-1\\0\end{bmatrix}$.

So $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \in \left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}\right\}$

But $f$ extends to a linear isometry $\tilde{f}$ of $\mathbb{R}^3$, so

$$f\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}\right) = f\left(-\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$
$$= -f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

We have 4 choices for $\begin{bmatrix}1\\0\\0\end{bmatrix}$,

but once a choice is made,

$f\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}\right)$ is determined by

this choice. This leaves

$f\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ precisely two

vectors to choose from, giving

$4 \cdot 2 = 8$ possible symmetries!