Symmetries and Matrices

(Section 1.4)

Recall: $\mathbb{R}^3$ with usual Euclidean distance

If $x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

with $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$,

then

$$\boxed{d((x_1,x_2,x_3),(y_1,y_2,y_3)) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}}$$

We'll continue to regard the square as occupying the position in the xy-plane that we worked with previously when computing products of symmetries.

Definition: (isometry) Given a region $R \subseteq \mathbb{R}^3$, and **isometry** of R is a map $f: R \to \mathbb{R}^3$ such that $\forall\, x, y \in R$,

$$\boxed{d(f(x), f(y)) = d(x, y)}$$

Theorem: (extension of isometries - 11.1.6)

Let $R \subseteq \mathbb{R}^3$ and let $f: R \to \mathbb{R}^3$ be an isometry. Then $\exists$ an extension $\tilde{f}: \mathbb{R}^3 \to \mathbb{R}^3$ such that $\tilde{f}|_R = f$, and

1) $\tilde{f}$ is an isometry

2) $\exists$ a linear map $A: \mathbb{R}^3 \to \mathbb{R}^3$ and $b \in \mathbb{R}^3$ such that

$$\boxed{\tilde{f}(x) = A(x) + b} \quad \forall\, x \in \mathbb{R}^3$$

Recall: (form of linear maps on $\mathbb{R}^n$)

A map $A: \mathbb{R}^n \to \mathbb{R}^m$ is linear (over $\mathbb{R}$) if $\forall\, x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

1) $A(x+y) = A(x) + A(y)$

2) $A(cx) = c \cdot A(x)$

Every such linear map may be represented as a matrix, also denoted by A.

The easiest matrix to obtain is the standard representation:

$1^{\underline{st}}$ column of matrix $= Ae_1$

$2^{\underline{nd}}$ column of matrix $= Ae_2$

$$\vdots$$

$n^{\underline{th}}$ column of matrix $= Ae_n$

where $\{e_1, e_2, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Idea: any isometry of $\mathbb{R}$ is "almost" a matrix - up to the additive constant b!

Definition: (centroid of polygon / polyhedra)

Given a polygon or a polyhedron R in $\mathbb{R}^3$, the **centroid** of R is the point $(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^3$

where

$$\bar{x} = \int_{\mathbb{R}^3} x \, \chi_R(x, y, z) \, dV$$

$$\bar{y} = \int_{\mathbb{R}^3} y \, \chi_R(x, y, z) \, dV$$

$$\bar{z} = \int_{\mathbb{R}^3} z \, \chi_R(x, y, z) \, dV$$

Here, $\chi_R : \mathbb{R}^3 \to \mathbb{R}$ is

the "characteristic" or

"indicator" function of R:

$$\chi_R(x,y,z) = \begin{cases} 1, & (x,y,z) \in R \\ 0, & (x,y,z) \notin R \end{cases}$$

Proposition: Any symmetry of a polygon or a polyhedron, with centroid located at the origin, extends to a linear isometry of $\mathbb{R}^3$

proof: Any symmetry of a polygon or a polyhedron preserves the polygon or polyhedron. So if R is the polygon or polyhedron and $f: R \to \mathbb{R}^3$ is the symmetry, then

$f(R) = R$. Then if the centroid of R is $(0,0,0)$, the centroid of $f(R)$ is $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ where

$$\bar{x}_1 = \int_{\mathbb{R}^3} x \, \chi_{f(R)}(x,y,z) \, dV$$

$$= \int_{\mathbb{R}^3} x \, \chi_R(x,y,z) \, dV$$

$$= \bar{x}$$

Similarly, $\bar{y}_1 = \bar{y}$ and $\bar{z}_1 = \bar{z}$.

Then f preserves the centroid, which is $(0,0,0)$. Every symmetry is an isometry, so extends to $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\tilde{f}(x,y,z) = A(x,y,z) + b$$

where A is a matrix.

But then

$$(0,0,0) = \tilde{f}(0,0,0) = A(0,0,0) + b$$
$$= (0,0,0) + b$$

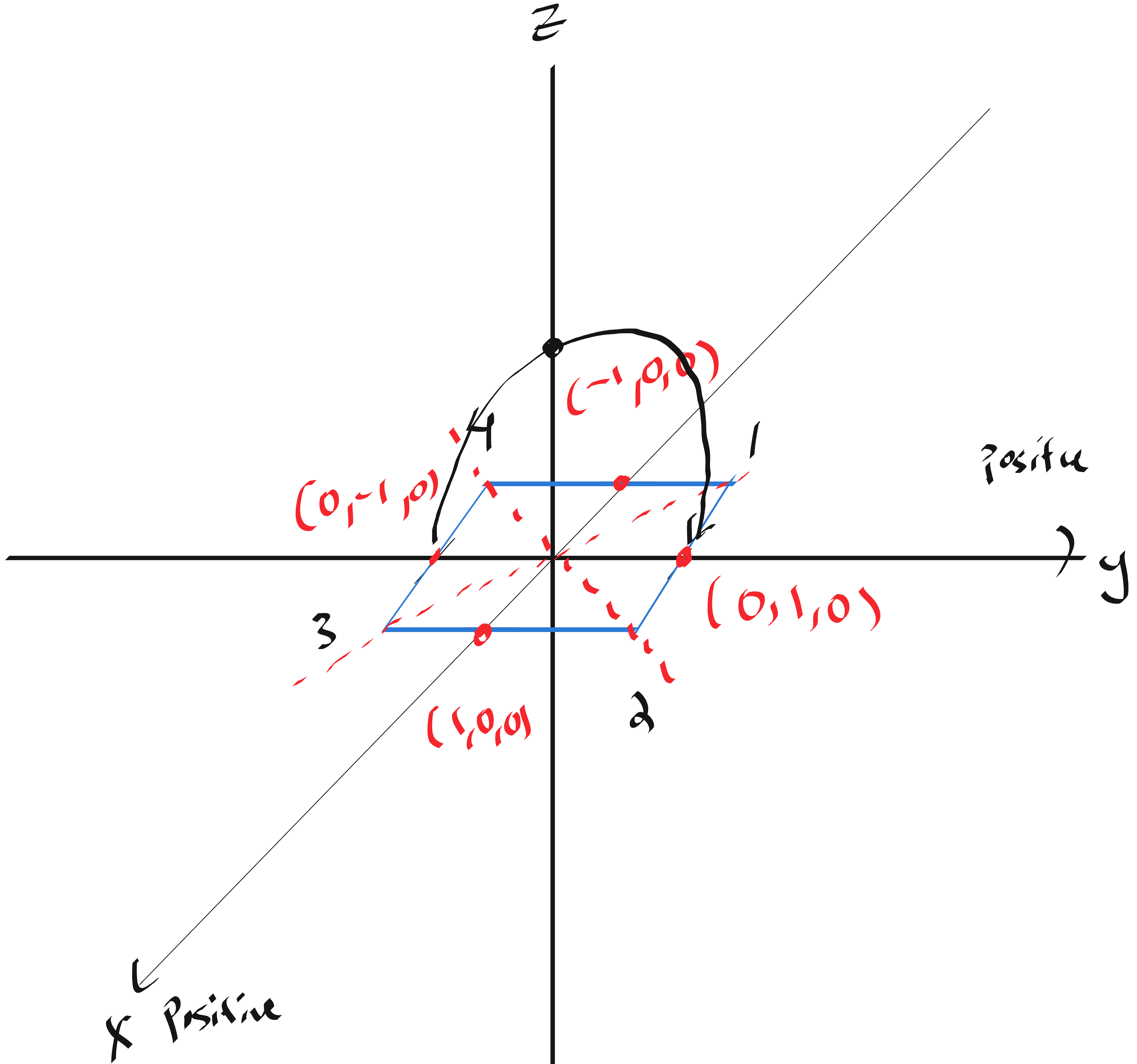
Therefore, $b = (0,0,0)$ and

$\tilde{f}$ is linear!

$\square$

Symmetries of the Square as Matrices

First: fix the square in the xy-plane



Matrix for r: (clockwise rotation by 90^0)

We just need to know where e_1, e_2, and e_3 go to under r.

$$r(e_3) = r\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$r(e_2) = r\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r(e_1) = r\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Matrix: $$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix for A:

$$A(e_3) = A\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A(e_2) = A\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$A(e_1) = A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Need to do B, C, and D...