Notation: $(M_n(k), GL_n(k))$

Let k be a field. We denote by $M_n(k)$ the $n \times n$ matrices with entries from k. $M_n(k)$ is a unital ring under the usual matrix addition and multiplication. $GL_n(k) \subseteq M_n(k)$ denotes all invertible matrices with respect to multiplication. So

$$GL_n(k) = M_n(k)^\times.$$

Example 3: (Heisenberg Group)

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\}$$

$$\subseteq GL_3(\mathbb{R})$$

Show G is a subgroup of $GL_3(\mathbb{R})$.

First, G is nonempty since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G.$$

Now let $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in G$.

We want to show that this matrix has an inverse in G.

Try to find $f, g, h \in \mathbb{R}$

such that

$$\underbrace{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & f & g \\ 0 & 1 & h \\ 0 & 0 & 1 \end{bmatrix}}_{\text{multiply out}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a+f & g+ah+b \\ 0 & 1 & h+c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

System of equations

$$\left.\begin{aligned} a+f &= 0 \\ g+ah+b &= 0 \\ h+c &= 0 \end{aligned}\right\} \quad f=-a, \quad h=-c$$

Plug $h=-c$ into the second equation:

$$g+a(-c)+b=0$$

$$g = ac - b$$

So

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Note: any matrix in G has determinant equal to 1, so will always be invertible.

(Here, the point is that

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

has the correct form.

Now if

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in G,$$

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a+x & y+az+b \\ 0 & 1 & c+z \\ 0 & 0 & 1 \end{bmatrix} \in G$$

By the Subgroup test, G is a subgroup of $GL_3(\mathbb{R})$.

Proposition: (one-step subgroup test)

Let G be a group with operation "$\cdot$". Let $H \subseteq G$ be nonempty. Then H is a subgroup of G if and only if $x \cdot y^{-1} \in H$ $\forall\, x, y \in H$.