Math 412/512 Practice Problems For Final

1) Let $R = \mathbb{Z} \times \mathbb{Z}$.

a) Describe all maximal ideals in R..

b) Determine all homomorphisms from R to $\mathbb{Z}$.

2) Let $S = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}\} \subset M_2(\mathbb{R})$. Is S a principal ideal domain?

3) Let R be a ring and let $x \in R$. Suppose that $\exists n \in \mathbb{N}$ such that $x^n = 0$ (note: $x^n = x \cdot x \cdot \cdots \cdot x$ n times). Prove that $1 - x$ is invertible.

4) Let R be a commutative ring with unity and $I \subset R$ an ideal. Recall that I is *prime* if for all $x, y \in R$, $xy \in I$ implies either $x \in I$ or $y \in I$. Prove that R/I is an integral domain if and only if I is prime.

5) Let F be a field. Prove that if $H \leq F^{\times}$ is a finite subgroup, then H is cyclic.

6) Suppose that G is a finite group and let p be the smallest prime dividing $|G|$. Let $H \leq G$ be a subgroup with $[G : H] = p$.

a) Prove that H is normal in G.

b) Prove that if G is abelian, $|G| = p(p+2)$ and both p, $p+2$ are primes, then G is cyclic.

7) Let R be an integral domain and let $a \in R$ be a nonzero, noninvertible element of R. Prove that $\langle a, x \rangle \subset R[x]$ is not a principal ideal domain.

8) Let G be a group with the property that for all $g, h, k \in G$, if $gh = kg$, then $h = k$. Prove that G is abelian.

9) Let $G = GL_2(\mathbb{R})$ and $H = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$. Show that $H \leq G$. Is H normal in G?

10) Recall that A_4 denotes all even permutations in S_4. Prove that $\mathcal{Z}(A_4) = e$ where e is the identity permutation.

11) Let G be an abelian group of odd order. Show that the product of all elements in G reduces to the identity. Find an example of a group with even order such that the product of all elements in the group is not the identity.

12) Given a group G, recall the definition of the commutator subgroup $[G, G] = \{ghg^{-1}h^{-1} : g, h \in G\}$. We have shown that $[G, G] \triangleleft G$. Prove that $G/[G, G]$ is abelian. This group is called the *abelianization* of G.

13) Let F be a field and $f \in F[x]$. Show that f has a multiple zero (or repeated root, if you prefer) in some extension E of F if and only if $f(x)$ and $f'(x)$ have a common factor of positive degree in $F[x]$. Here, $f'(x)$ refers to the ordinary derivative.

14) Consider $p(x) = x^4 + x^2 + 1 \in \mathbb{Q}[x]$. Find the splitting field of p.