Math 451/551 Assignment 1

Due Monday, January 24

1) a) If S is a set, prove that $(S^c)^c = S$.

b) If S and T are sets, show that if $S \subseteq T$, then $T^c \subseteq S^c$. Either prove the converse is true or give a counterexample.

2) (Exercise 1.2.10) Let $y_1 = 1$ and for each $n \in \mathbb{N}$, define $y_{n+1} = \dfrac{3y_n + 4}{4}$.

a) Use induction to prove that the sequence satisfies $y_n < 4$ for all $n \in \mathbb{N}$.

b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is increasing.

3) (Exercise 1.2.7) Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D \ : \ f(x) \in B\}$. This set is called the *preimage* of B.

a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$ in this case? Does $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B)$?

b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B)$ and $g^{-1}(A) \cup g^{-1}(B) = g^{-1}(A \cup B)$ for all sets $A, B \subseteq \mathbb{R}$.

4) Let $S \subseteq \mathbb{R}$ and define

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

The function χ_S is usually called the *characteristic* or indicator function for S.

a) Determine $\chi_S^{-1}([0, 1))$, $\chi_S^{-1}((0, 1))$, $\chi_S^{-1}((0, 1])$ and $\chi_S^{-1}([0, 1])$.

b) If T is an arbitrary subset of $\mathbb{R}$, characterize $\chi_S^{-1}(T)$, with proof.

c) Prove that, for all subsets S and T of $\mathbb{R}$, $\chi_{S\cap T} = \chi_S \cdot \chi_T$ and $\chi_{S\cup T} = \chi_S + \chi_T - \chi_{S\cap T}$

5) Let $S, T \subseteq \mathbb{R}$. Suppose $\sup(S)$ and $\sup(T)$ both exist and that neither S nor T is empty.

a) Prove that $\inf(S) = -\sup(-S)$ where $-S = \{-x \mid x \in S\}$.

b) Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

c) Is there a statement similar to b) for $\sup(S \cap T)$? Justify your reasoning.