Math 451/551 Assignment 1

Due Thursday, September 20

1) Let S, T, and A be sets.

a) Prove that $A \cup (S \cap T) = (A \cup S) \cap (A \cup T)$.

b) If S^c denotes the complement of a set S, prove that

$$A \cap (S \cup T)^c = (A \cap S^c) \cap (A \cap T^c).$$

2) a) Let $n, k \in \mathbb{N} \cup \{0\}$, $0 \leq k \leq n$, and define $\left(\begin{array}{c} n \\ k \end{array} \right) = \dfrac{n!}{k!(n-k)!}$. Determine the value of $\displaystyle\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)$, then use induction to prove that your guess is correct.

b) (Exercise 1.2.11) Use induction (or any other method you prefer) to show that if a set A contains n elements, then the number of different subsets of A is equal to 2^n. (Keep in mind that the empty set $\emptyset$ is considered to be a subset of every set.)

3) Given a function $f : S \to T$ and a subset $A \subseteq T$, let $f^{-1}(A)$ be the set of all points from the domain S that get mapped into A; that is, $f^{-1}(A) = \{x \in S \ : \ f(x) \in A\}$. This set is called the *preimage* of A.

a) Demonstrate that for any such map f and $B \subseteq ran(f)$, $f(f^{-1}(B)) = B$. Is it true that $f^{-1}(f(A)) = A$ for all $A \subseteq S$? Prove or give a counterexample.

b) (Exercise 1.2.7, b)) Show that for an arbitrary function $g : S \to T$, it is always true that $g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B)$ and $g^{-1}(A) \cup g^{-1}(B) = g^{-1}(A \cup B)$ for all sets $A, B \subseteq T$. Convince yourself (but don't prove it unless you want to) that the same results holds for infinite unions and intersections.

4) Let $S \subseteq \mathbb{R}$ be nonempty and define

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

The function χ_S is usually called the *characteristic* or indicator function for S.

a) Determine $\chi_S^{-1}(\mathbb{Q})$, $\chi_S^{-1}((0,\infty))$, $\chi_S^{-1}([0,1))$ and $\chi_S^{-1}([0,1])$.

b) If T is an arbitrary subset of $\mathbb{R}$, determine $\chi_S^{-1}(T)$, with proof. *Hint*: your answer will depend on the elements in T.

c) Prove that, for all nonempty subsets S and T of $\mathbb{R}$, $\chi_{S\cap T} = \chi_S \cdot \chi_T$ and $\chi_{S\cup T} = \chi_S + \chi_T - \chi_{S\cap T}$

5) Let $S, T \subseteq \mathbb{R}$. Suppose neither S nor T is empty and that $\sup(S)$ and $\sup(T)$ both exist.

a) Prove that $\inf(S) = -\sup(-S)$ where $-S = \{-x \mid x \in S\}$.

b) Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

c) Is there a statement similar to b) for $\sup(S \cap T)$? Justify your reasoning.