Math 451/551 Assignment 2

Due Tuesday, February 1

1) (Exercise 1.2.5) Let x and y be real numbers. Prove that $||x|-|y|| \leq |x-y|$.

2) A real number α is said to be *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n$ and a polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ with $p(\alpha) = 0$. Any real number that is not algebraic is said to be *transcendental*. It is by no means obvious that transcendental numbers exist, but both π (Lindemann, 1880) and e (Hermite, 1873) are transcendental.

a) It is a standard, though technical, result in Abstract Algebra that the sum of two algebraic numbers is algebraic. Assuming this result, show that if α is transcendental and β is algebraic, then $\alpha + \beta$ is transcendental.

b) (Exercise 1.4.12) Fix $n \in \mathbb{N}$ and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable. (*Hint*: for each $m \in \mathbb{N}$, consider polynomials $\sum_{i=0}^{n} a_i x^i$ that satisfy $\sum_{i=0}^{n} |a_i| \leq m$.)

c) (Exercise 1.4.12) Argue that the set of all algebraic numbers is countable.

d) Check that the algebraic numbers are dense in $\mathbb{R}$.

e) (Extra Credit, worth an A in the course) Prove that $\pi + e$ is transcendental.

3) Consider $S = \{x \in \mathbb{R} : x^3 < 2\}$. S is bounded above, so S has a least upper bound. Call this number β.

a) By using the proof of Theorem 1.4.5 as a guide, show that $\beta = \sqrt[3]{2}$.

b) Prove that $\sqrt[3]{2}$ is irrational.

c) Prove that $\sqrt[3]{2} + \sqrt{6}$ is irrational.

4) a) Demonstrate that $[0, 1]$ and $(0, 1)$ have the same cardinality.

b) Show that if x, y, a, and b are real numbers with $x < y$ and $a < b$, then the cardinality of $[a, b]$ equals the cardinality of $[x, y]$.

c) Conclude that any two finite intervals have the same cardinality.

d) (Extra Credit) Show that $\mathbb{R}^2$ and $\mathbb{R}$ have the same cardinality. *Note:* I will accept no written solution to this problem. If you want credit, you have to explain your solution to me in my office.