Math 451/551 Assignment 3

Due Thursday, February 10

1) Show that the map $\phi : (0,1) \to \mathbb{R}$ given in class by

$$\phi(x) = \begin{cases} \frac{1}{x} - 2 & 0 < x \leq \frac{1}{2} \\ 2 + \frac{1}{x-1} & \frac{1}{2} < x < 1 \end{cases}$$

is a bijection.

2) (Exercise 2.3.5, variant) a) Show that if $(b_n)_{n\in\mathbb{N}}$ converges to b, then the sequence of absolute values $(|b_n|)_{n\in\mathbb{N}}$ converges to $|b|$.

b) Is the converse of part a) true? That is, if we know that $(|b_n|)_{n\in\mathbb{N}}$ converges, can we then deduce $(b_n)_{n\in\mathbb{N}}$ converges?

3)(continued fractions) Let $b_1 = 1$ and $b_n = 1 + \dfrac{1}{1 + b_{n-1}}$ for all $n \geq 2$. Note that $b_n \geq 1$ for all $n \in \mathbb{N}$.

a) Show that $b_{2k+1}^2 < 2$ for all $k \in \mathbb{N}$.

b) Show that $b_{2k}^2 > 2$ for all $k \in \mathbb{N}$.

c) Prove that $(b_{2k+1})_{k\in\mathbb{N}}$ is increasing. Conclude that $(b_{2k+1})_{k\in\mathbb{N}}$ converges.

d) Prove that $(b_{2k})_{k\in\mathbb{N}}$ is decreasing. Conclude that $(b_{2k})_{k\in\mathbb{N}}$ converges.

e) (Extra Credit) Determine, with proof, that $(b_{2k})_{k\in\mathbb{N}}$ and $(b_{2k+1})_{k\in\mathbb{N}}$ both converge to the same number. Hence, $(b_n)_{n\in\mathbb{N}}$ converges to that number. Once more, I will accept no written solution. You must present your solution to me in my office.

4) (Exercise 2.4.6, variant) Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence.

a) Prove that the sequence defined for all $n \in \mathbb{N}$ by $y_n = \sup\{a_k : k \geq n\}$ converges.

The *limit superior* of $(a_n)_{n\in\mathbb{N}}$, or $\limsup_{n\to\infty} a_n$, is then defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n.$$

where $(y_n)_{n\in\mathbb{N}}$ is the sequence from part (a) of this exercise.

b) If $n \in \mathbb{N}$ and $z_n = \inf\{a_k : k \geq n\}$, we define the *limit inferior* of $(a_n)_{n\in\mathbb{N}}$, or $\liminf_{n\to\infty}(a_n)$, by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n.$$

As in a), it can be shown that the limit exists. For the sequence

$$a_n = (-1)^n,$$

determine $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence $(a_n)_{n\in\mathbb{N}}$. Observe that part b) gives an example of a sequence that exhibits strict inequality.

d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim_{n\to\infty} a_n$ exists. In this case, all three share the same value.

Extra Credit: Prove that $|\mathbb{R}| = |2^{\mathbb{Z}}|$. I will accept no written solution. You must present your solution to me in my office.