Math 451/551 Assignment 4

Due Tuesday, February 21

1) (Exercise 2.7.5) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be Cauchy sequences of real numbers.

a) Show, without appealing to the Cauchy Criterion, that if $z_n = x_n + y_n$, then $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers.

b) Do the same for $w_n = x_n \cdot y_n$.

2) (Constructing the real numbers) Let C be the space of all Cauchy sequences of rational numbers. Define an equivalence relation on C by, if $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in C$,

$$x \sim y \text{ iff } \lim_{n\to\infty} |x_n - y_n| = 0.$$

a) Prove that this is an equivalence relation on C.

Let X be the space whose elements are equivalence classes of elements in C. If $x \in C$, let $[x]$ denote the resulting point in X.

b) Let $q \in \mathbb{Q}$ and consider the Cauchy sequence $Q \in C$,

$$Q = (q)_{n\in\mathbb{N}}.$$

Show that the map $\phi : \mathbb{Q} \to X$ given by $\phi(q) = [Q]$ is an injection.

c) Define, for elements $[x], [y] \in X$,

$$[x] \cdot [y] = [(x_n \cdot y_n)_{n\in\mathbb{N}}],$$

$$[x] + [y] = [(x_n + y_n)_{n\in\mathbb{N}}].$$

By Exercise #1, the term-wise sum or product of Cauchy sequences is again Cauchy, and so these operations define elements in X. Check that both these operations are well-defined.

It then follows from associativity and commutativity of multiplication on rational numbers that both operations are commutative and associative. Distributivity of "$\cdot$" over "+" follows similarly.

d) Consider $\phi(0), \phi(1) \in X$. Show that for all $[x] \in X$, there exists $[y] \in X$ with $[x]+[y] = \phi(0)$. Similarly, show that for all $[x] \in X$, $[x] \neq \phi(0)$, there exists $[z] \in X$, $[x] \cdot [z] = \phi(1)$. Determine explicitly $[y]$ and $[z]$ if $[x] = \phi(q)$ for some $q \in \mathbb{Q}$, $q \neq 0$.

e) If $[x], [y] \in X$, define a partial ordering "$\leq$" by $[x] \leq [y]$ if and only if for all $q \in \mathbb{Q}$, $q > 0$, there exists $N \in \mathbb{N}$ such that $x_n - q \leq y_n$ for all $n \geq N$. Show that this partial ordering is well-defined.

Now the hard part: we want to prove that X satisfies the Completeness Axiom of the real number system: if $S \subset X$ is nonempty and bounded above (there exists $[x] \in X$ such that $[x] \geq [y]$ for all $[y] \in S$), then S has a least upper bound. The proof will be outlined in the next three steps.

f) Argue that there exists an upper bound of the form $\phi(q)$ for some $q \in \mathbb{Q}$ and a $p \in Q$ such that $\phi(p) < [z]$ for at least one $[z] \in S$.

g) With p and q as in part f), define sequences of rationals $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ by

$$x_1 = q, \ y_1 = p,$$

and set $z_1 = (x_1+y_1)/2$. If $\phi(z_1)$ is an upper bound for S, set $x_2 = z_1$ and $y_2 = y_1$. If $\phi(z_1)$ is not an upper bound for S, set $x_2 = x_1$ and $y_2 = z_1$. Continuing in this manner, assume x_n and y_n are chosen and let $z_n = (x_n + y_n)/2$. If $\phi(z_n)$ is an upper bound for S, set $x_{n+1} = z_n$ and $y_{n+1} = y_n$. If $\phi(z_n)$ is not an upper bound for S, set $x_{n+1} = x_n$ and $y_{n+1} = z_n$. Prove x and y are Cauchy sequences and hence in C.

h) Let x and y be as in part g). For all $n \in \mathbb{N}$, show via induction that $\phi(x_n)$ is an upper bound for S and that $\phi(y_n)$ is not an upper bound for S. Use this information to conclude that $[x]$ is the least upper bound of S. Are you now ready to call X by its proper name?

3) If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in a metric space X with metric d, show that the sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ converges. *Hint:* For any n, m, we have by the triangle inequality

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n).$$

Now subtract $d(x_m, y_m)$ from both sides.

EXTRA CREDIT: All notation is as in Exercise # 2. Again, no written solutions are permitted, but must be justified in office hours.

If $[x], [y] \in X$, define

$$D([x],[y]) = \lim_{n\to\infty} |x_n - y_n|.$$

By Exercise #3, this limit exists.

a) Show that $D([x],[y])$ is well-defined; that is, it is unchanged if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are replaced by equivalent sequences.

b) Prove that D, as given in part b), is a metric on X.

c) Prove that $\phi(\mathbb{Q})$ is dense in X, i.e., for all $\varepsilon > 0$ and $[x] \in X$, there exists a $q \in Q$ with $D([x], \phi(q)) < \varepsilon$.