Math 451/551 Assignment 5

Due Tuesday, March 27

1) Determine, with proof, which of the following sets are open, closed, or neither in $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$.

a) The irrational numbers.

b) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

c) The algebraic numbers.

2) (Exercise #3.2.9) a) Prove that if $A, B \subseteq \mathbb{R}$, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$ with the usual metric.

b) Does the result in a) extend to infinite unions? Prove or give a counterexample.

3) (Exercise #3.3.4) Prove that if $K \subseteq \mathbb{R}$ is compact in the usual metric and $F \subseteq \mathbb{R}$ is closed, then $F \cap K$ is compact.

4) (Closed-Set Version of Compactness) A collection $\mathcal{S} = \{S_i\}_{i \in \mathcal{I}}$ of subsets of a metric space X is said to have the *finite-intersection property* if for all $S_{i_1}, S_{i_2}, \ldots, S_{i_n} \in \mathcal{S}$, $\cap_{k=1}^{n} S_{i_k} \neq \emptyset$.

a) Let X be metric space. Let $\mathcal{S} = \{S_i\}_{i \in \mathcal{I}}$ be a collection of *closed* subsets of X with $X = \cup_{i \in \mathcal{I}} S_i$. Show that X is compact if and only if for every such collection $\mathcal{S}$ with the finite intersection property,

$$\bigcap_{i \in \mathcal{I}} S_i \neq \emptyset.$$

b) Find a collection $\mathcal{S} = \{S_i\}_{i \in \mathcal{I}}$ of closed subsets of $\mathbb{R}$ in the usual metric satisfying the finite-intersection property, but $\cap_{i \in \mathcal{I}} S_i = \emptyset$.

5) Recall we showed that if X is any set, the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

for all $x, y \in X$. Let's give this a name and call it the *discrete metric.* Now suppose $X = \mathbb{N}$.

a) Determine $B(1, 1)$ and $B(1, 2)$ under the discrete metric.

b) Characterize the compact subsets of $\mathbb{N}$ under the discrete metric, with proof.

EXTRA CREDIT:

The Heine-Borel theorem can be badly false in an infinite dimensional vector space. Let X be the space of all bounded sequences of real numbers. Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ be two such sequences and define

$$d(x, y) = \sup_{n\in\mathbb{N}} |x_n - y_n|.$$

a) Prove that this is a metric on X.

b) Prove that X with this metric is complete.

c) Let $\mathbf{0} = (0)_{n\in\mathbb{N}}$. Show that $\overline{B(\mathbf{0}, 1)}$ is not compact, though it is closed and bounded.