Math 451/551 Assignment 6

Due Thursday, April 5

1) a) Prove that if a subset S of $\mathbb{R}$ with the usual metric is both compact and connected, then S is a closed interval (possibly degenerate).

b) (Exercise 3.4.7) If $S \subseteq \mathbb{R}$ with the usual metric is connected, is $\overline{S}$ connected?

2) (Exercise 3.4.9) A set S in a metric space X is called *totally disconnected* if for any $x, y \in S$, there exist separated sets A and B with $x \in A$, $y \in B$ and $S = A \cup B$. Let $C = \cap_{n=1}^{\infty} C_n$ be the Cantor set. Refer to the description given in class for the notation, or switch to the book's less precise, though more intuitive, definition.

a) Given $x, y \in C$ with $x < y$, set $\varepsilon = y - x$. For each $n \in \mathbb{N}$, C_n consists of a finite union of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval in C_N.

b) Argue that for all $x, y \in C$, there exists $z \notin C$ with $x < z < y$. Explain how this proves that there can be no interval of the form (a, b) with $a < b$ contained in C.

c) Show that C is totally disconnected.

3) (Exercise 4.3.2) Prove rigorously that $\lim_{x \to 1} x^{1/3} = 1$. *Hint*: the identity $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ may be helpful.

4) a) Let f and g be two continuous functions on $\mathbb{R}$ with the usual metric and let $S \subset \mathbb{R}$ be countable. Show that if $f(x) = g(x)$ for all $x \in S^c$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

b) Let f and g be continuous functions from a metric space X into a metric space Y. Let S be a dense subset of X. Prove that $f(S)$ is dense in $f(X)$ and that if $f(s) = g(s)$ for all $s \in S$, then $f(x) = g(x)$ for all $x \in X$.

5) Characterize all continuous functions from $\mathbb{N}$ with the discrete metric into $\mathbb{R}$ with the usual metric.